Chapter 5: Linear Transformations

1. Let $T : M_{33} \to M_{33}$ be given by

$$T(\mathbf{X}) = \frac{1}{2}(\mathbf{AX} + \mathbf{XA})$$

Show that *T* is a linear transformation.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by

$$T(\mathbf{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } T(\mathbf{e}_2) = \begin{pmatrix} -b \\ a \end{pmatrix}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis for $\mathbb{R}^2$ and $a^2 + b^2 = 1$.

(i) Find the angle between $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.

(ii) Determine the matrix which represents *T*.

(iii) Determine the inverse transformation T^{-1}.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear operator given by $T(\mathbf{x}) = \mathbf{Ax}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & -1 & -2 \\ 5 & -3 & -1 & 1 \end{pmatrix}$$

Show that $\mathbf{x} = \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}^T$ is in the kernel of *T*.

Find a basis for the kernel of *T* given that $\operatorname{rank}(T) = 3$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(\mathbf{x}) = \mathbf{Ax} \text{ where } \mathbf{A} = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

Let $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the four vectors of the unit square.

Find the image of the unit square under the given transformation for $r = 2,\ \theta = 45°$, and sketch this on the plane. What effect does the matrix **A** have on the unit square?

5. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\mathbf{u}_i) = \mathbf{Au}_i \text{ where } \mathbf{A} \text{ is a 2 by 2 matrix}$$

Find the matrix **A** which defines the transformation

$$T(\mathbf{u}_i) = \mathbf{v}_i \text{ for } i = 1,\ 2,\ 3 \text{ for the vectors shown in Fig. 1}$$

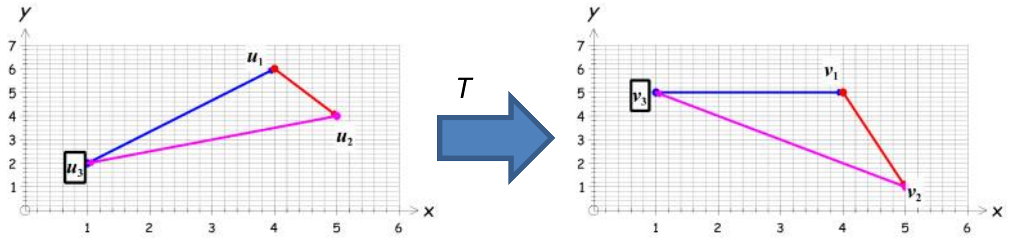

Figure 1

6. Consider the linear transformation $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\mathbf{u}_i) = \mathbf{A}\mathbf{u}_i$$

where **A** is a 2 by 2 matrix.

Find the matrix **A** which defines the transformation

$T(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1,\ 2,\ 3$ for the vectors shown in Fig. 2

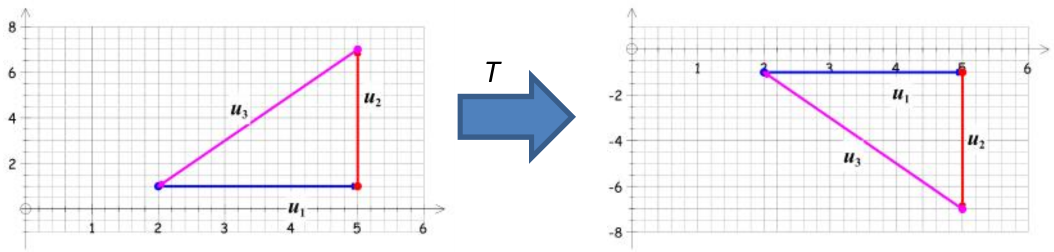

Figure 2

7. Let $T:\ U \to V$ be a linear transform given by

$T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where **A** is a 2 by 3 matrix and **x** is a column vector

Determine the vector spaces *U* and *V*.

8. Let $T:\ U \to V$ be the linear transform given by

$T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where **A** is a 2 by 2 matrix and **x** is a column vector

Given that linear transform switches the *x* and *y* coordinates such as:

$$T\left[\begin{pmatrix}1\\2\end{pmatrix}\right] = \begin{pmatrix}2\\1\end{pmatrix} \text{ and } T\left[\begin{pmatrix}1\\3\end{pmatrix}\right] = \begin{pmatrix}3\\1\end{pmatrix}$$

find the matrix **A**.

9. (i) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator which scales any given vector by 3:

$$T(\mathbf{x}) = 3\mathbf{x}$$

Find the matrix which represents *T*.

(ii) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator which scales any given vector by *n*:

$$T(\mathbf{x}) = n\mathbf{x}$$

Find the matrix which represents *T*.

(iii) Find the inverse transformation T^{-1} in each case and explain what the inverse transformation does to an input vector **x**.

10. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator which satisfies

$$T\left[\begin{pmatrix}1\\2\end{pmatrix}\right]=3\begin{pmatrix}1\\2\end{pmatrix} \text{ and } T\left[\begin{pmatrix}1\\4\end{pmatrix}\right]=-2\begin{pmatrix}1\\4\end{pmatrix}$$

Find the matrix **A** which represents the transformation *T*.

11. Let $T:\ U \to V$ be a linear transform given by

$T(\mathbf{x})=\mathbf{Ax}$ where **A** is a 3 by 2 matrix and **x** is a column vector

Given that $T\left[\begin{pmatrix}1\\0\end{pmatrix}\right]=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $T\left[\begin{pmatrix}0\\1\end{pmatrix}\right]=\begin{pmatrix}4\\5\\6\end{pmatrix}$ determine the matrix **A** and the kernel of *T*.

12. Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by

$T\left[\begin{pmatrix}x\\y\end{pmatrix}\right]=x\mathbf{e}_1+y\mathbf{e}_2$ where $\mathbf{e}_1,\ \mathbf{e}_2$ are the standard unit vectors of $\mathbb{R}^2$

A circle or an ellipse in $\mathbb{R}^2$ can be described by vectors given by $T\begin{pmatrix}a\cos(\text{¯})\\b\sin(\text{¯})\end{pmatrix}$.

Sketch the circle or ellipse for the following values of *a* and *b*:

(a) $a=b=1$ (b) $a=b=2$ (c) $a=5,\ b=2$

13. Show that $\mathbf{u}=\begin{pmatrix}\cos(\text{¯})\\\sin(\text{¯})\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}-\sin(\text{¯})\\\cos(\text{¯})\end{pmatrix}$ are orthonormal vectors in $\mathbb{R}^2$.

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ by a linear operator given by $T(\mathbf{x})=\mathbf{Ax}$. Determine whether $T(\mathbf{u})$ and $T(\mathbf{v})$ are orthonormal vectors for the following matrices:

(a) $\mathbf{A}=\begin{pmatrix}3&0\\0&3\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}3&0\\0&2\end{pmatrix}$

In each case sketch **u**, **v**, $T(\mathbf{u})$ and $T(\mathbf{v})$ for any value of ¯ you like.

14. In computer graphics we deal with various transformations.
Consider the unit square shown in the figure below:

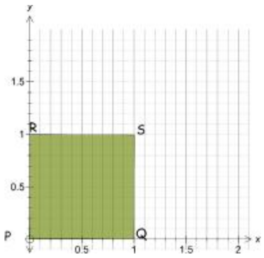

Figure 3

Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

Determine the image of the unit square under the transformation *T* where

(a) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

In each case describe and sketch the effect of the transformation.

15. In *computer graphics* the following transformation is important:

$$T(\mathbf{x}) = M\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\mathbf{x} + \mathbf{y}$$

Where *M* is a scalar, **x** and **y** are vectors in $\mathbb{R}^2$.

Consider the unit square shown in the figure below:

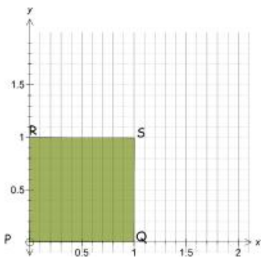

Figure 4

Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined above by $T(\mathbf{x})$.

For the following values of *M* and vector **y** find the image of the unit square shown in Fig. 4 for $\theta = 45°$.

(a) $M = 4,\ \mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (b) $M = 4,\ \mathbf{y} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (c) $M = \frac{1}{2},\ \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

In each case describe and sketch the effect of the transformation on the unit square.

16. Let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator given by

$$T(\mathbf{u}) = \mathbf{M}_k\mathbf{u}$$

In *computer graphics* the following matrices are important:

(a) $\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$ (b) $\mathbf{M}_2 = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$

(c) $\mathbf{M}_3 = \begin{pmatrix} \cos(\text{¯}) & -\sin(\text{¯}) & 0 \\ \sin(\text{¯}) & \cos(\text{¯}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Taking $\text{¯} = 90°$ and $\mathbf{u} = (1 \quad 1 \quad 1)^T$ determine $T(\mathbf{u})$ for each of the above matrices. Also sketch **u** and $T(\mathbf{u})$ for each matrix and describe the effect of applying the linear operator.

Show that each of the matrices $\mathbf{M}_1,\ \mathbf{M}_2$ and $\mathbf{M}_3$ are orthogonal and determine the inverse transform $T^{-1}(\mathbf{x})$.

17. Let $T:\ \mathbb{R}^n \to \mathbb{R}$ be a linear transform given by

$$T(\mathbf{u}) = \mathbf{x} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u}$$

Prove that $\mathbf{x} \neq \mathbf{y}$.

18. Let $T:\ \mathbb{R}^n \to \mathbb{R}$ be a linear transform given by

$$T(\mathbf{u}) = \mathbf{x} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} \text{ for every vector } \mathbf{u} \text{ in } \mathbb{R}^n$$

Prove that $\mathbf{x} = \mathbf{y}$.

19. Let $T:\ \mathbb{R}^3 \to \mathbb{R}$ by given by

$$T\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = x + y + z$$

Show that *T* is a linear transformation and find a basis for the kernel of *T*.

20. Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be a linear operator. Find the rank and nullity of *T* such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for each of the following cases:

(a) $\mathbf{A} = \begin{pmatrix} -2 & 5 & 7 & 13 \\ 6 & -15 & -21 & -39 \\ 14 & -35 & -49 & -91 \\ 2 & -5 & -7 & -13 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & -1 & 5 & 1 \\ 5 & 2 & 1 & 1 \\ 4 & 13 & 3 & 20 \end{pmatrix}$

Also determine a basis for $\ker(T)$ in each case.

21. Let $T:\ \mathbb{R}^7 \to \mathbb{R}^4$ be the linear transform given by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}=\begin{pmatrix}1&2&3&0&1&2&0\\0&1&4&0&2&4&0\\0&0&0&2&3&6&0\\0&0&0&0&0&8&3\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7\end{pmatrix}^T$.

Find a basis for the kernel of *T*.

(Hint: Consider the columns of matrix **A**).

22. Let $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix}$ and $\mathbf{u}=\begin{pmatrix}a\\b\end{pmatrix}, \mathbf{v}=\begin{pmatrix}-b\\a\end{pmatrix}$ where $a\neq 0,\ b\neq 0$ are normalized vectors. Determine the inverse transformation $T^{-1}(\mathbf{x})$.

23. Show that $T\left[\begin{pmatrix}x\\y\end{pmatrix}\right]=\begin{pmatrix}x+a\\y+b\end{pmatrix}$ where $a\neq 0$ or $b\neq 0$ is *not* a linear transformation. If $a=b=0$ then what transformation do we have?

24. A linear transformation $T:\ \mathbb{R}^4\to\mathbb{R}^4$ relative to the standard basis $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_4)$ is represented by

$$\mathbf{A}=\begin{pmatrix}1&2&0&1\\3&0&-1&2\\2&5&3&1\\1&2&1&3\end{pmatrix}$$

Find the matrix of the transformation relative to the basis

(a) $(\mathbf{e}_1,\ \mathbf{e}_3,\ \mathbf{e}_2,\ \mathbf{e}_4)$ (b) $(\mathbf{e}_1,\ \mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

25. Let the linear transform $T:\ P_1\to P_2$ be defined by

$$T(ax+b)=a\left(1+x+x^2\right)+b$$

Determine the matrix **A** which represents the given transformation with respect to the ordered basis $B=(1,\ x)$ and $C=\left(1,\ x,\ x^2\right)$ for P_1 and P_2 respectively. Find $T(x+2)$ by using this matrix.

[Remember P_n is the vector space of polynomials of degree *n* or less.]

26. Let *V* be the vector space $V=span\left\{\sin(x)\cos(x),\ \sin^2(x),\ \cos^2(x)\right\}$ and $T:V\to V$ be the linear transform given by

$$T(f)=f' \text{ where } f' \text{ is the derivative of } f$$

Find the matrix representation of *T* with respect to the basis

$$B=\left\{\sin(x)\cos(x),\ \sin^2(x),\ \cos^2(x)\right\}$$

Determine $f'(x)$ of $f(x) = -5\sin(x)\cos(x) + 6\sin^2(x) - 2\cos^2(x)$ by using this matrix.

27. Let $T : V \to V$ be a linear operator. Let **u** be a vector such that $T^n(\mathbf{u}) \neq \mathbf{O}$ where n is a natural number. Prove that the vectors

$$\mathbf{u},\ T(\mathbf{u}),\ T^2(\mathbf{u}),\ T^3(\mathbf{u}),\ \cdots,\ T^{n-1}(\mathbf{u})$$

are non-zero.

Brief Solutions to Problems of Chapter 5

2. (i) $90°$ (ii) $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (iii) $T^{-1}(\mathbf{x}) = \begin{pmatrix} ax + by \\ -bx + ay \end{pmatrix}$
3. $\mathbf{x} = \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}^T$
5. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -8/5 & 6/5 \end{pmatrix}$
6. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
7. $U = \mathbb{R}^3,\ V = \mathbb{R}^2$
8. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
9. (i) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ (iii) $T^{-1}(\mathbf{x}) = \frac{1}{3}\mathbf{x}$, $T^{-1}(\mathbf{x}) = \frac{1}{n}\mathbf{x}$ provided $n \neq 0$
10. $\mathbf{A} = \begin{pmatrix} 8 & -5/2 \\ 20 & -7 \end{pmatrix}$
11. $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$, $\{\mathbf{O}\}$
13. (a) $T(\mathbf{u})$ and $T(\mathbf{v})$ are orthogonal but not normalized (b) Not orthogonal
14. (a) Rotate about the origin through $90°$ anti-clockwise.
    (b) Reflection in the line $y = -x$.
    (c) Scaled by a factor of 3.
    (d) Reflected in the line $y = x$ and scaled by a factor of 3.
15. In each case the square has been rotated by $45°$ in an anti-clockwise direction with the centre the origin. Then
    (a) Enlarged by a scale factor of 4 and shifted to the right by 2 units and up by 1 unit.
    (b) Enlarged by a scale factor of 4 and shifted to the left by 2 units and up by 1 unit.
    (c) Enlarged by a scale factor of 1/2 and shifted to the right and up by 1 unit.

16. (a) $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $T^{-1}(\mathbf{x}) = \begin{pmatrix} x \\ y\cos(\text{¯}) + z\sin(\text{¯}) \\ -y\sin(\text{¯}) + z\cos(\text{¯}) \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $T^{-1}(\mathbf{x}) = \begin{pmatrix} x\cos(\text{¯}) - z\sin(\text{¯}) \\ y \\ x\sin(\text{¯}) + z\cos(\text{¯}) \end{pmatrix}$

(c) $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $T^{-1}(\mathbf{x}) = \begin{pmatrix} x\cos(\text{¯}) + y\sin(\text{¯}) \\ -x\sin(\text{¯}) + y\cos(\text{¯}) \\ z \end{pmatrix}$

19. $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

20. (a) $rank(T) = 1$, $\text{nullity}(T) = 3$, a basis for kernel is $\left\{ \begin{pmatrix} 5/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 13/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $\text{rank}(T) = 3$, $\text{nullity}(T) = 1$, a basis for kernel is $\left\{ \begin{pmatrix} -53 \\ 145 \\ 69 \\ -94 \end{pmatrix} \right\}$

21. $\left\{ \begin{pmatrix} -18 \\ 12 \\ 0 \\ 9 \\ 0 \\ -3 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ -4 \\ 0 \\ -3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

22. $T^{-1}\left[\begin{pmatrix} x \\ y \end{pmatrix} \right] = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

23. Identity transformation

24. (a) $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 & 3 & 4 \\ 3 & 3 & 2 & 4 \\ 2 & 7 & 10 & 11 \\ 1 & 3 & 4 & 7 \end{pmatrix}$

25. $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$, $3 + x + x^2$

26. $\mathbf{A}=\begin{pmatrix} 0 & 2 & -2 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $16\sin(x)\cos(x)+5\sin^2(x)-5\cos^2(x)$