Solutions to Problems on Chapter 2 (Only available to tutors.)

1. Since the vectors are multiples of each so they are linearly dependent:

$$\begin{pmatrix}1\\2\\3\end{pmatrix},\ \begin{pmatrix}-1\\-2\\-3\end{pmatrix},\ \begin{pmatrix}-3\\-6\\-9\end{pmatrix}$$

2. (a) Vectors **u** and **v** do not lie along the same line so they cannot be multiples of each other. Hence **u** and **v** are linearly independent.
(b) Vectors **u** and **v** do lie along the same line so they are linearly dependent.

3. Consider the linear combination:

$$k_1\mathbf{v}_1 \quad + \quad k_2\mathbf{v}_2 \quad + \quad k_3\mathbf{v}_3 \quad + \quad \cdots \quad + \quad k_n\mathbf{v}_n \quad = \quad \mathbf{O}$$

We can write this in matrix form as $\mathbf{Ax}=\mathbf{O}$. We are given that the reduced row echelon form has *n* leading ones so we have

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{pmatrix} 1 & & \mathbf{0} & \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix} \end{array} \begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix} \quad \Rightarrow \quad k_1 = k_2 = \cdots = k_n = 0$$

The reduced row echelon form is the identity matrix.
By chapter 2 we have the following:

> Theorem (2.24). Let $\mathbf{A}$ be an *n* by *n* matrix, then the following 2 statements are equivalent:
> (a) The reduced row echelon form of the matrix **A** is the identity matrix **I**.
> (b) Columns of matrix **A** are linearly independent.

Since the columns of matrix **A** are the vectors in the given set $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ so these are linearly independent by part (b). By the following result:

> Proposition (2.27). Any *n* linearly independent vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

We conclude that the set of vectors $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ form a basis for $\mathbb{R}^n$.

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4. It is enough to show that the given vectors are linearly independent:

$$x\begin{pmatrix}2\\-3\\1\end{pmatrix}+y\begin{pmatrix}0\\1\\2\end{pmatrix}+z\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

From the first row we have

$2x+z=0 \quad \Rightarrow \quad z=-2x=-2r$ where $x=r$ is a real number. By using the middle row we have

$$-3x+y+z=0 \quad \Rightarrow \quad y=3x-z=3r-(-2r)=5r$$

Substituting $x=r,\ y=5r$ and $z=-2r$ into the bottom equation

$$x+2y-2z=r+2(5r)-2(-2r)=15r=0 \quad \Rightarrow \quad r=0$$

We have $x=y=z=0$ so the given vectors $\mathbf{u},\ \mathbf{v}$ and $\mathbf{w}$ are linearly independent so they form a basis for $\mathbb{R}^3$.

We need to find scalars $x,\ y,\ z$ such that

$$x\mathbf{u}+y\mathbf{v}+z\mathbf{w}=\begin{pmatrix} a & b & c \end{pmatrix}^T$$

Writing the given vectors in an augmented matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 2 & 0 & 1 & a \\ -3 & 1 & 1 & b \\ 1 & 2 & -2 & c \end{array}\right)$$

Carrying out the following row operations:

$$\begin{matrix} R_1 \\ R_2-R_1 \\ R_3+2R_1 \end{matrix}\left(\begin{array}{ccc|c} 2 & 0 & 1 & a \\ -5 & 1 & 0 & b-a \\ 5 & 2 & 0 & c+2a \end{array}\right)$$

Adding the bottom two rows gives

$$\begin{matrix} x & y & z & \end{matrix}$$
$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & a \\ -5 & 1 & 0 & b-a \\ 0 & 3 & 0 & c+a+b \end{array}\right)$$

By the bottom row we have $3y=a+b+c \quad \Rightarrow \quad y=\dfrac{a+b+c}{3}$.

From the middle row we have

$$-5x+y=b-a \quad \Rightarrow \quad 5x=y-b+a$$

Substituting $y=\dfrac{a+b+c}{3}$ into this yields

$$5x=\frac{a+b+c}{3}-b+a=\frac{a+b+c-3b+3a}{3}=\frac{4a-2b+c}{3}$$
$$x=\frac{4a-2b+c}{15}$$

From the top row we have $2x+z=a$. Substituting $x=\dfrac{4a-2b+c}{15}$ into this

$$2\left(\frac{4a-2b+c}{15}\right)+z=a \quad \Rightarrow \quad z=a-2\left(\frac{4a-2b+c}{15}\right)$$

$$=\frac{15a-8a+4b-2c}{15}=\frac{7a+4b-2c}{15}$$

Our scalars are $x=\frac{4a-2b+c}{15}$, $y=\frac{a+b+c}{3}$ and $z=\frac{7a+4b-2c}{15}$:

$$\left(\frac{4a-2b+c}{15}\right)\begin{pmatrix}2\\-3\\1\end{pmatrix}+\left(\frac{a+b+c}{3}\right)\begin{pmatrix}0\\1\\2\end{pmatrix}+\left(\frac{7a+4b-2c}{15}\right)\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

5. We are given that vectors **u** and **v** are parallel. This means that

$\mathbf{u}=m\mathbf{v}$ where *m* is a non-zero scalar

Why?

Because **u** and **v** are parallel so they are in the same direction but may have different lengths. We apply the following result of chapter 2:

(2.12)	$\cos(¯)=\frac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\lVert\mathbf{v}\rVert}$	provided **u** and **v** are non-zero vectors.

Substituting $\mathbf{u}=m\mathbf{v}$ into this (2.12) gives

$$\cos(¯)=\frac{(m\mathbf{v})\cdot\mathbf{v}}{\lVert m\mathbf{v}\rVert\lVert\mathbf{v}\rVert}$$

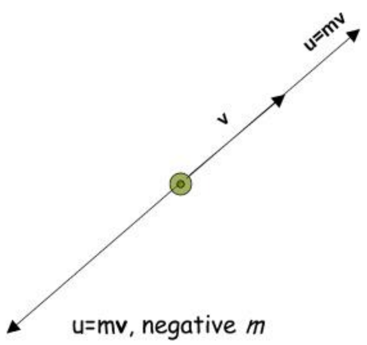
$$=\frac{m(\mathbf{v}\cdot\mathbf{v})}{\lVert m\rVert\lVert\mathbf{v}\rVert^2}=\frac{m\lVert\mathbf{v}\rVert^2}{\lVert m\rVert\lVert\mathbf{v}\rVert^2}=\frac{m}{\lVert m\rVert}=\pm 1$$

Therefore we have our required result; $\cos(¯)=\pm 1$.

Taking inverse cosine we have

$$¯=\cos^{-1}(\pm 1)=0 \text{ or } 180°$$

This means that the given vectors are in the same direction or out of phase by 180°. We have



6. By the above (2.12) we have $r = \cos(\theta) = \dfrac{\mathbf{h}\cdot\mathbf{w}}{\|\mathbf{h}\|\|\mathbf{w}\|}$. Evaluating each of the components gives:

$$\mathbf{h}\cdot\mathbf{w} = \begin{pmatrix}1.7\\1.8\\1.9\end{pmatrix}\cdot\begin{pmatrix}75\\70\\100\end{pmatrix} = (1.7\times 75)+(1.8\times 70)+(1.9\times 100) = 443.5$$

$$\|\mathbf{h}\| = \left\|\begin{pmatrix}1.7\\1.8\\1.9\end{pmatrix}\right\| = \sqrt{1.7^2+1.8^2+1.9^2} = 3.12; \quad \|\mathbf{w}\| = \left\|\begin{pmatrix}75\\70\\100\end{pmatrix}\right\| = \sqrt{75^2+70^2+100^2} = 143.27$$

Substituting this into the formula gives

$$r = \cos(\theta) = \frac{443.5}{3.12\times 143.27} = 0.321$$

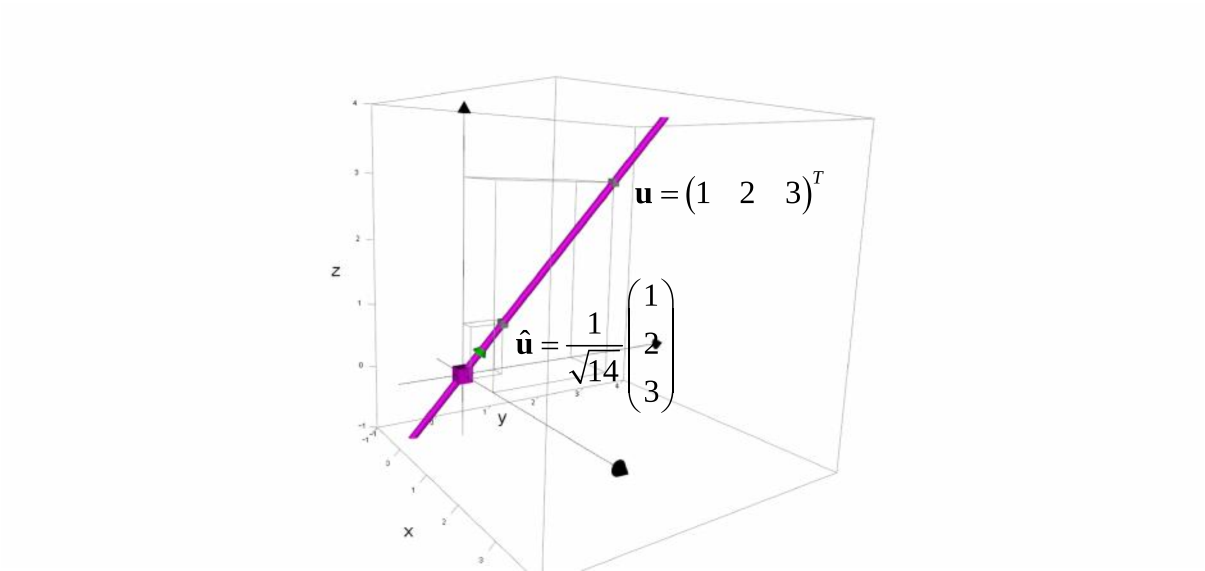
Hence there is a positive correlation between height and weight as we expected.

7. Carrying out the dot product evaluation

$$\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}\sin(A)\\\cos(A)\end{pmatrix}\cdot\begin{pmatrix}\cos(B)\\\sin(B)\end{pmatrix} = \sin(A)\cos(B)+\cos(A)\sin(B)$$

$$= \sin(A+B) \qquad [\text{Using a trigonometric identity}]$$

8. Since the space is spanned by one vector in $\mathbb{R}^3$ so the space *S* is just a line through the point $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$ as shown below:



A unit vector in the space *S* is

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{1^2+2^2+3^2}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

9. *How do we show that the given vectors* $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ *form an orthogonal basis for* $\mathbb{R}^3$*?*

We prove that each of the vectors $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ are orthogonal to each other:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-4\end{pmatrix}=2+2-4=0,\ \mathbf{u}\cdot\mathbf{w}=\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix}=3-4+1=0,\ \mathbf{v}\cdot\mathbf{w}=\begin{pmatrix}2\\1\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix}=6-2-4=0$$

The given vectors are orthogonal to each other. *How do show that these form a basis for* $\mathbb{R}^3$*?*

They $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ form a basis for $\mathbb{R}^3$ because

(2.32) Any n non-zero orthogonal vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

We need to write the vector $\mathbf{x}=\begin{pmatrix}7\\1\\9\end{pmatrix}$ as a linear combination of $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$:

$$\mathbf{x}=\begin{pmatrix}7\\1\\9\end{pmatrix}=k_1\begin{pmatrix}1\\2\\1\end{pmatrix}+k_2\begin{pmatrix}2\\1\\-4\end{pmatrix}+k_3\begin{pmatrix}3\\-2\\1\end{pmatrix}$$

Using row operations to solve this:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\left(\begin{array}{ccc|c}1&2&3&7\\2&1&-2&1\\1&-4&1&9\end{array}\right)$$

Carrying out the following row operations:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^\dagger=\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3^\dagger=\mathrm{R}_3-\mathrm{R}_1\end{matrix}\left(\begin{array}{ccc|c}1&2&3&7\\0&-3&-8&-13\\0&-6&-2&2\end{array}\right)$$

The final row operation is

$$\begin{matrix}\\\mathrm{R}_1\\\mathrm{R}_2^\dagger\\\mathrm{R}_3^\dagger-2\mathrm{R}_2^\dagger\end{matrix}\begin{matrix}\begin{matrix}k_1&k_2&k_3\end{matrix}\\\left(\begin{array}{ccc|c}1&2&3&7\\0&-3&-8&-13\\0&0&14&28\end{array}\right)\end{matrix}$$

From the bottom row we have $k_3=2$. Substituting this into the middle row yields

$$-3k_2-8k_3=-13\ \Rightarrow\ -3k_2-8(2)=-13\ \Rightarrow\ -3k_2=3\ \Rightarrow\ k_2=-1$$

Substituting these $k_2 = -1$ and $k_3 = 2$ into the top row gives

$$k_1 + 2k_2 + 3k_3 = 7 \quad \Rightarrow \quad k_1 + 2(-1) + 3(2) = 7 \quad \Rightarrow \quad k_1 = 7 + 2 - 6 = 3$$

Hence $\mathbf{x} = \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ or in compact form $\mathbf{x} = 3\mathbf{u} - \mathbf{v} + 2\mathbf{w}$.

10. We write each of the given vectors in a matrix:

$$\mathbf{A} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 6 & 9 & 11 & 15 \end{pmatrix}$$

By using Matlab or Maple we can place this matrix into reduced row echelon form:

$$\begin{pmatrix} \boxed{1} & 0 & -1 & 0 & -3 \\ 0 & \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{pmatrix}$$

The entries which are boxed are the leading ones. Hence these columns are linearly independent so they form a basis for $\mathbb{R}^3$. Hence $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\}$ form a basis for $\mathbb{R}^3$.

11. For the vector **u** to be orthogonal we need to ensure that the dot product between **u**, **v** and **u**, **w** is zero. Let $\mathbf{u} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ then

$$\mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2y + 3z = 0, \quad \mathbf{w} \cdot \mathbf{u} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x + 5y + 6z = 0$$

Writing these equations in an augmented matrix we have

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right)$$

Carrying out the following row operation:

$$\begin{matrix} \\ R_1 \\ R_2 - 4R_1 \end{matrix} \begin{matrix} \begin{matrix} x & y & z \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right) \end{matrix}$$

From the bottom row we have

$$-3(y + 2z) = 0 \quad \Rightarrow \quad y + 2z = 0 \quad \Rightarrow \quad y = -2z$$

Let $z = 1$ then $y = -2$. Substituting these into the top row $x + 2y + 3z = 0$ gives

$$x + 2(-2) + 3(1) = 0 \quad \Rightarrow \quad x = 1$$

Our orthogonal vector is

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Remember we need to find the unit vector in this direction. Normalizing the vector

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \left[\text{Because } \|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}\right]$$

Since vectors $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ are orthogonal so they form a basis for $\mathbb{R}^3$. *Why?*

Because

> Proposition (2.32). Any *n* non-zero orthogonal vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

12. Using the following definition:

> (2.9) $\quad d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

We have

$$\begin{aligned} d(\mathbf{u}+\mathbf{v},\quad \mathbf{u}-\mathbf{v}) &= \|\mathbf{u}+\mathbf{v}-(\mathbf{u}-\mathbf{v})\| \\ &= \|2\mathbf{v}\| \end{aligned}$$

Using the following result:

> Proposition (2.10). Let **u** be a vector in $\mathbb{R}^n$ and *k* be a real scalar. Then
>
> (ii) $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$

We have $d(\mathbf{u}+\mathbf{v},\quad \mathbf{u}-\mathbf{v}) = \|2\mathbf{v}\| = |2|\|\mathbf{v}\| = 2\|\mathbf{v}\|$. This is our required result.

13. Let $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $k = 3$. Then

$$\mathbf{u} + k\mathbf{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

Therefore

$$(\mathbf{u}+k\mathbf{v})\cdot(\mathbf{u}+k\mathbf{v}) = 2\begin{pmatrix} 1 \\ 6 \end{pmatrix}\cdot 2\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 4\left[1^2 + 6^2\right] = 148$$

Evaluating the RHS of $(\mathbf{u}+k\mathbf{v})\cdot(\mathbf{u}+k\mathbf{v}) = \|\mathbf{u}\|^2 + 2k(\mathbf{u}\cdot\mathbf{v}) + k\|\mathbf{v}\|^2$ gives

$$\begin{aligned} \|\mathbf{u}\|^2 + 2k(\mathbf{u}\cdot\mathbf{v}) + k\|\mathbf{v}\|^2 &= \left\|\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\|^2 + \left(2\times 3\times\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 4 \end{pmatrix}\right]\right) + 3\left\|\begin{pmatrix} 0 \\ 4 \end{pmatrix}\right\|^2 \\ &= 4 + 0 + 3(16) = 52 \end{aligned}$$

Since the last result, 52, does not equal the penultimate result, 148, so the given result is false.

14. Expanding the Left Hand Side gives

$$\begin{aligned}(k\mathbf{u}+c\mathbf{v})\cdot(k\mathbf{u}-c\mathbf{v}) &= (k\mathbf{u}\cdot k\mathbf{u})+(k\mathbf{u}\cdot(-c\mathbf{v}))+(c\mathbf{v}\cdot k\mathbf{u})-(c\mathbf{v}\cdot c\mathbf{v})\\ &= k^2(\mathbf{u}\cdot\mathbf{u})\underbrace{-kc(\mathbf{u}\cdot\mathbf{v})+ck(\mathbf{u}\cdot\mathbf{v})}_{=0}-c^2(\mathbf{v}\cdot\mathbf{v})\\ &= k^2\|\mathbf{u}\|^2-c^2\|\mathbf{v}\|^2\end{aligned}$$

Since the vectors **u** and **v** are unit vectors so they have a length or norm of 1 which means that $\|\mathbf{u}\|^2=\|\mathbf{v}\|^2=1$. Substituting this into the above gives

$$(k\mathbf{u}+c\mathbf{v})\cdot(k\mathbf{u}-c\mathbf{v})=k^2\|\mathbf{u}\|^2-c^2\|\mathbf{v}\|^2=k^2-c^2$$

This completes our proof.

15. (a) We need to see if a linear combination of the columns of matrix **A** give us **b**:

$$x\begin{pmatrix}1\\0\\5\end{pmatrix}+y\begin{pmatrix}1\\3\\-7\end{pmatrix}+z\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}15\\11\\11\end{pmatrix}$$

We augment the matrix **A** with the vector **b**:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\left(\begin{array}{ccc|c}1&1&2&15\\0&3&1&11\\5&-7&2&11\end{array}\right)$$

Carrying out the row operation

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3^*=\mathrm{R}_3-5\mathrm{R}_1\end{matrix}\left(\begin{array}{ccc|c}1&1&2&15\\0&3&1&11\\0&-12&-8&-64\end{array}\right)$$

Executing the row operation

$$\begin{matrix}\\\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3^*+4\mathrm{R}_2\end{matrix}\begin{matrix}\begin{matrix}x&y&z&\end{matrix}\\\left(\begin{array}{ccc|c}1&1&2&15\\0&3&1&11\\0&0&-4&-20\end{array}\right)\end{matrix}$$

From the bottom row we have

$$4z=20\quad\Rightarrow\quad z=5$$

Substituting this into the middle row gives

$$3y+z=11\quad\Rightarrow\quad 3y+5=11\quad\Rightarrow\quad y=2$$

Putting $y=2$ and $z=5$ into the top row gives

$$x+y+2z=15\quad\Rightarrow\quad x+2+10=15\quad\Rightarrow\quad x=3$$

We have $x=3$, $y=2$ and $z=5$ therefore a linear combination of the columns of matrix **A** give us the vector **b**, so **b** is in the space spanned by the columns of **A**.
(b) Similarly we have

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & 3 & 9 & 2 \\ 5 & -7 & -11 & 3 \end{array}\right)$$

Carrying out the row operation:

$$\begin{matrix} R_1 \\ R_2 \\ R_3^* = R_3 - 5R_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & 3 & 9 & 1 \\ 0 & -12 & -36 & -2 \end{array}\right)$$

Executing

$$\begin{matrix} R_1 \\ R_2 \\ R_3^* + 4R_2 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & 3 & 9 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

This system is inconsistent because in the bottom row we have

$$0 = 2$$

Hence the vector **b** does not lie in the space spanned by the columns of matrix **A**.

16. Place the given set of vectors $\{\mathbf{x},\ \mathbf{x}+\mathbf{e}_1,\ \mathbf{x}+\mathbf{e}_2\}$ in a matrix:

$$\mathbf{A} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}+\mathbf{e}_1 \\ \mathbf{x}+\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1+1 & x_2 & x_3 & x_4 \\ x_1 & x_2+1 & x_3 & x_4 \end{pmatrix}$$

Labelling the rows:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1+1 & x_2 & x_3 & x_4 \\ x_1 & x_2+1 & x_3 & x_4 \end{pmatrix}$$

Executing the following row operations:

$$\begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Hence for this to be linearly independent we must have either $x_3 \neq 0$ and $x_4 \neq 0$.

17. Consider the linear combination

$$x\mathbf{u}+y\mathbf{v}+z\mathbf{w}=\mathbf{O}$$

$$x\begin{pmatrix}k\\2\\1\end{pmatrix}+y\begin{pmatrix}2\\k\\4\end{pmatrix}+z\begin{pmatrix}k\\12\\6\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{bmatrix}\text{Substituting the given values of}\\ \text{the vectors } \mathbf{u},\ \mathbf{v} \text{ and } \mathbf{w}\end{bmatrix}$$

The augmented matrix is

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}k&2&k&0\\2&k&12&0\\1&4&6&0\end{array}\right)$$

Carrying out the row operation:

$$\begin{matrix}\\R_1\\R_2-2R_3\\R_3\end{matrix}\begin{matrix}\begin{matrix}x&y&z\end{matrix}\\ \left(\begin{array}{ccc|c}k&2&k&0\\0&k-8&0&0\\1&4&6&0\end{array}\right)\end{matrix}$$

By expanding the middle row we have

$$(k-8)y=0$$

For linear independence all our scalars must be zero so $y=0$ which means that $k-8\neq 0$ or $k\neq 8$.

Substituting $y=0$ into the top row gives

$$k(x+z)=0 \;\Rightarrow\; x+z=0 \;\Rightarrow\; x=-z \qquad (*)$$

Putting this into the bottom row gives

$$-z+6z=5z=0 \;\Rightarrow\; z=0$$

Hence $x=0$. From (*) we must have $k\neq 0$ because if $k=0$ then x and *z* can be any values but for linear independence all the scalars *x*, *y* and *z* must be zero.

Hence we have linear independence provided $k\neq 0$ and $k\neq 8$.

18. For orthogonality the dot product of the two vectors must be zero:

$$\begin{pmatrix}k\\3\\2\end{pmatrix}\cdot\begin{pmatrix}k\\k\\1\end{pmatrix}=k^2+3k+2=0 \;\Rightarrow\; (k+2)(k+1)=0 \;\Rightarrow\; k=-2,\ -1$$

We need to find a third vector which is orthogonal to these two vectors. We first find the vector $\mathbf{u}=\begin{pmatrix}x & y & z\end{pmatrix}^T$ which is orthogonal to the vectors in *S* for $k=-2$:

$$\begin{pmatrix}-2\\3\\2\end{pmatrix},\begin{pmatrix}-2\\-2\\1\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Finding the dot product between each of the first two vectors with the last:

$$\begin{pmatrix}-2\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=-2x+3y+2z=0$$

$$\begin{pmatrix}-2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=-2x-2y+z=0 \quad\Rightarrow\quad z=2x+2y$$

Substituting $z=2x+2y$ into the first equation gives:

$$-2x+3y+2(2x+2y)=2x+7y=0 \quad\Rightarrow\quad x=-\frac{7}{2}y$$

Let $y=2$ then $x=-7$. Putting these into $z=2x+2y$ gives

$$z=2(-7)+2(2)=-10$$

The vector $\mathbf{u}=\begin{pmatrix}-7\\2\\-10\end{pmatrix}$.

Let $\mathbf{v}=\begin{pmatrix}x' & y' & z'\end{pmatrix}^T$ be the vector which is orthogonal to the vectors in *S* for $k=-1$:

$$\begin{pmatrix}-1\\3\\2\end{pmatrix},\begin{pmatrix}-1\\-1\\1\end{pmatrix},\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$

Finding the dot product between each of the first two vectors with the last:

$$\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=-x'+3y'+2z'=0 \quad\Rightarrow\quad x'=3y'+2z'$$

$$\begin{pmatrix}-1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=-x'-y'+z'=0$$

Substituting $x'=3y'+2z'$ into the bottom equation yields:

$$-(3y'+2z')-y'+z'=-4y'-z'=0 \quad\Rightarrow\quad z'=-4y'$$

Let $y'=1$ then $z'=-4$. Putting these into $x'=3y'+2z'$ gives

$$x'=3(1)+2(-4)=-5$$

The vector $\mathbf{v}=\begin{pmatrix}-5\\1\\-4\end{pmatrix}$.

19. (a) *How do we determine whether the two given vectors are linear dependent?*
Check that they are multiples of each other. Since

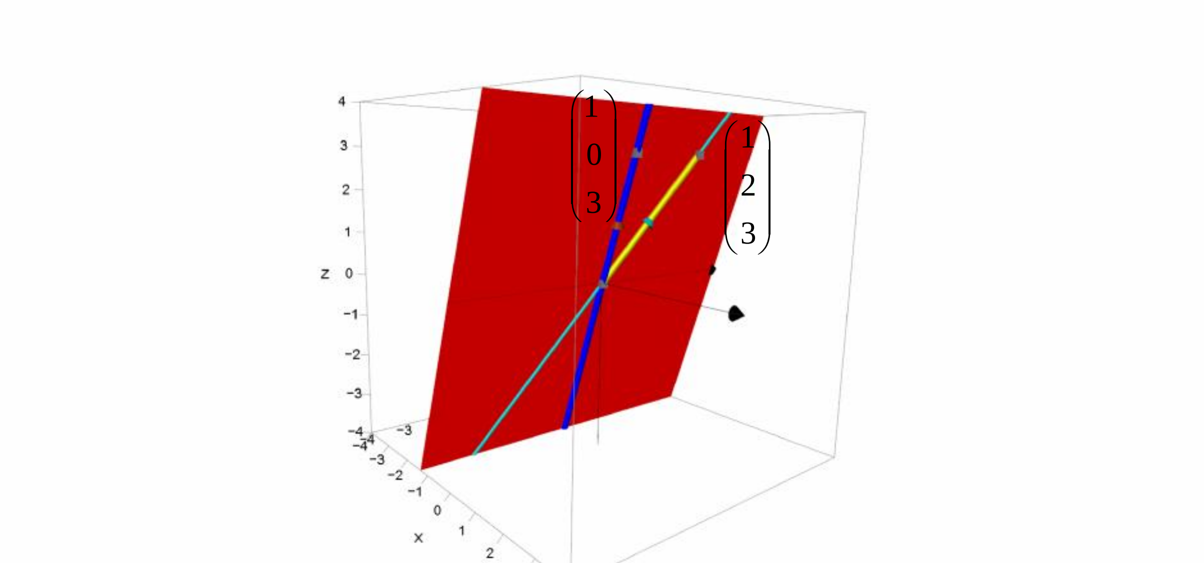
$$\mathbf{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}\neq m\begin{pmatrix}1\\0\\3\end{pmatrix}$$

So **u** and **v** are linearly independent.

(b) The space spanned by $\{\mathbf{u},\ \mathbf{v}\}$ is given by

$$k\mathbf{u}+c\mathbf{v}=k\begin{pmatrix}1\\2\\3\end{pmatrix}+c\begin{pmatrix}1\\0\\3\end{pmatrix}$$

where *k* and *c* are scalars. As these two vectors are linearly independent so they span a plane in $\mathbb{R}^3$ as shown below:



We can write the space spanned as *S* where

$$S=\left\{k\mathbf{u}+c\mathbf{v}\ \middle|\ \mathbf{u}=\begin{pmatrix}1\\2\\3\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}1\\0\\3\end{pmatrix}\right\}$$

(c) Vector $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ needs to be linearly independent of the vectors **u** and **v**. Any vector which is not in the plane shown above will do. For example a vector on the *x* – axis will do, so $\mathbf{w}=\begin{pmatrix}1\\0\\0\end{pmatrix}$.

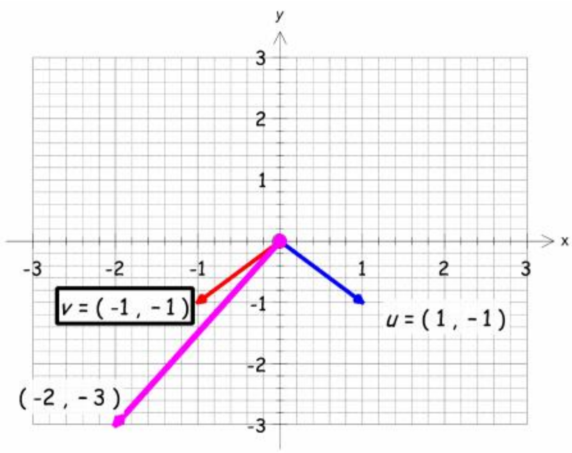
20. For $\mathbb{R}^2$ we need 2 basis vectors. It is enough to can show that the two given vectors are linearly independent in order to form a basis. The given vectors

$$\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix}\neq m\mathbf{v}=m\begin{pmatrix}-1\\-1\end{pmatrix}$$

are not multiples of each other so they are linearly independent. Hence

$$B=\left\{\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}-1\\-1\end{pmatrix}\right\}$$

forms a basis for $\mathbb{R}^2$.



We need to find the scalars, *k* and *c*, in the following linear combination:

$$k\mathbf{u}+c\mathbf{v}=k\begin{pmatrix}1\\-1\end{pmatrix}+c\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}-2\\-3\end{pmatrix}$$

Solving this for *k* and *c* gives $k=\frac{1}{2},\ c=\frac{5}{2}$. Therefore

$$\mathbf{w}=\begin{pmatrix}-2\\-3\end{pmatrix}=\frac{1}{2}\mathbf{u}+\frac{5}{2}\mathbf{v}$$

21. Clearly the given vectors **u** and **v** are linearly independent because they are not multiples of each other;

$$\begin{pmatrix}2\\3\end{pmatrix}\neq m\begin{pmatrix}0\\4\end{pmatrix}$$

We have 2 linearly independent vectors in $\mathbb{R}^2$ so we have a basis for $\mathbb{R}^2$. Consider the linear combination

$$k\begin{pmatrix}2\\3\end{pmatrix}+c\begin{pmatrix}0\\4\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix} \quad \Rightarrow \quad k=\frac{1}{2},\quad c=\frac{1}{8}$$

We have $\frac{1}{2}\mathbf{u}+\frac{1}{8}\mathbf{v}=\begin{pmatrix}1\\2\end{pmatrix}$ or the coordinates with respect to *B* are $\begin{pmatrix}1/2\\1/8\end{pmatrix}_B$.

22. Using the formula of chapter 2:

(2.12)	$\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\lVert\mathbf{v}\rVert}$

We have $\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=-1$ and

$$\lVert\mathbf{u}\rVert=\left\lVert\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}\right\rVert=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=2,\qquad \lVert\mathbf{v}\rVert=\left\lVert\begin{pmatrix}0\\1\end{pmatrix}\right\rVert=1$$

Substituting this into (2.12) gives

$$\cos(\theta)=\frac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\lVert\mathbf{v}\rVert}=-\frac{1}{2}$$

Hence $\theta=\cos^{-1}\left(-\frac{1}{2}\right)=120^\circ$.

Since the angle between the vectors is not equal to zero or 180° so they are not multiples of each other which means they are linearly independent. Therefore vectors **u** and **v** form a basis for $\mathbb{R}^2$.

*How do we write the vector $\begin{pmatrix}1 & 1\end{pmatrix}^T$ in terms of **u** and **v**?*

By finding the scalars in the linear combination:

$$k\mathbf{u}+c\mathbf{v}=k\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}+c\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \quad \Rightarrow \quad k=\frac{1}{\sqrt{3}},\quad c=\frac{\sqrt{3}+1}{\sqrt{3}}$$

The coordinates with respect to the given basis *B* is

$$\begin{pmatrix}1/\sqrt{3}\\ \left(\sqrt{3}+1\right)/\sqrt{3}\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1+\sqrt{3}\end{pmatrix}_B$$

23. (a) We need to find the shortest distance between the straight line $y=2x+1$ and the vector $\mathbf{u}=\begin{pmatrix}3 & 4\end{pmatrix}^T$.

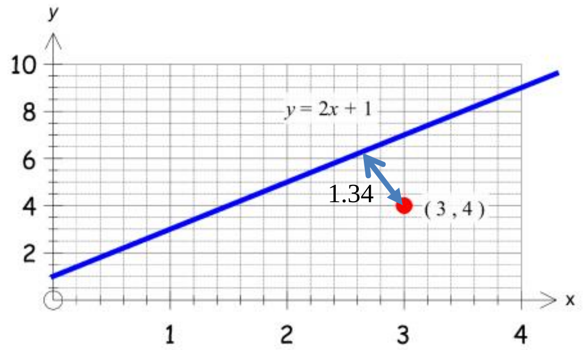
We can write $y=2x+1$ as $y-2x-1=0$ and in dot product form this is represented by

$$\underbrace{\begin{pmatrix}-2\\1\end{pmatrix}}_{=\mathbf{v}}\cdot\underbrace{\begin{pmatrix}x\\y\end{pmatrix}}_{=\mathbf{x}}+\underbrace{(-1)}_{=c}=0$$

Using the given formula $\dfrac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}$ with $\mathbf{u}=(3\quad 4)^T$:

$$\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}=\frac{\left|\begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix}-1\right|}{\left\|\begin{pmatrix}-2\\1\end{pmatrix}\right\|}=\frac{|-6+4-1|}{\sqrt{(-2)^2+1^2}}=\frac{3}{\sqrt{5}}=1.34\text{ (3sf)}$$

Sketching this we have



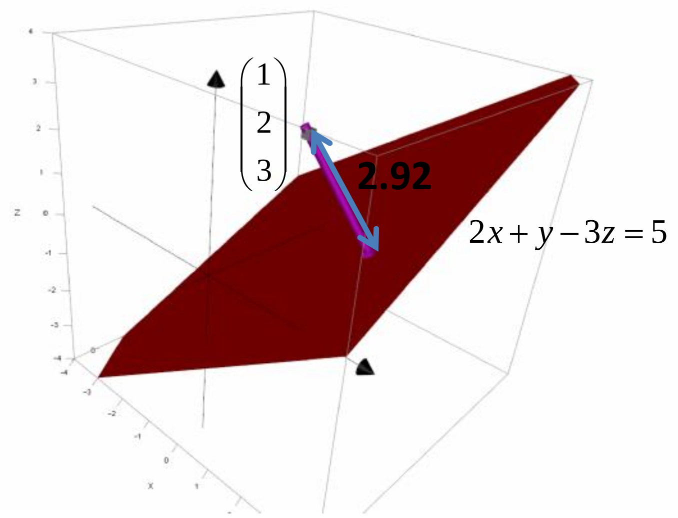
(b) Writing the given plane $2x+y-5z=5$ or $2x+y-5z-5=0$ in dot product form:

$$2x+y-5z-5=\underbrace{\begin{pmatrix}2\\1\\-5\end{pmatrix}}_{=\mathbf{v}}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}+\underbrace{(-5)}_{=c}=0$$

Using the given formula $\dfrac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}$ with $\mathbf{u}=(1\quad 2\quad 3)^T$:

$$\frac{|\mathbf{u}\cdot\mathbf{v}+c|}{\|\mathbf{v}\|}=\frac{\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-5\end{pmatrix}-5\right|}{\left\|\begin{pmatrix}2\\1\\-5\end{pmatrix}\right\|}=\frac{|2+2-15-5|}{\sqrt{2^2+1^2+(-5)^2}}=\frac{16}{\sqrt{30}}=2.92\text{ (3sf)}$$

Sketching this we have



24. We need to use the angle between vector formula of chapter 2:

(2.12)	$\cos(\theta)=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\lVert\mathbf{v}\rVert}$

(a) We have

$$\cos(\theta)=\frac{\mathbf{q}\cdot\mathbf{d}}{\lVert\mathbf{q}\rVert\lVert\mathbf{d}\rVert}=\frac{\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\3\end{pmatrix}}{\left\lVert\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\right\rVert\left\lVert\begin{pmatrix}1\\1\\1\\3\end{pmatrix}\right\rVert}=\frac{0.25[1+1+1+3]}{\sqrt{4\times0.25^2}\sqrt{(3\times1^2)+3^2}}=\frac{3/2}{\sqrt{12}/2}=\frac{3}{\sqrt{12}}$$

Therefore $\theta=\cos^{-1}\left(\dfrac{3}{\sqrt{12}}\right)=30^\circ$.

(b) Similarly for $\mathbf{d}=\begin{pmatrix}1 & 0 & 1 & 1\end{pmatrix}^T$ we have

$$\cos(\theta)=\frac{\mathbf{q}\cdot\mathbf{d}}{\lVert\mathbf{q}\rVert\lVert\mathbf{d}\rVert}=\frac{\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\1\end{pmatrix}}{\left\lVert\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\right\rVert\left\lVert\begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\rVert}=\frac{0.25(3)}{\sqrt{4\times0.25^2}\sqrt{3\times1^2}}=\frac{0.75}{\sqrt{3}}=0.433$$

The angle $\theta=\cos^{-1}(0.433)=64.34^\circ$.

(c) For $\mathbf{d}=\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T$ we have

$$\cos(\bar{\ }) = \frac{\mathbf{q}\cdot\mathbf{d}}{\|\mathbf{q}\|\|\mathbf{d}\|} = \frac{\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}{\left\|\begin{pmatrix}0.25\\0.25\\0.25\\0.25\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\|} = \frac{0.25[1+2+3+4]}{\sqrt{4\times 0.25^2}\sqrt{1^2+2^2+3^2+4^2}} = \frac{5/2}{\sqrt{30}/2} = \frac{5}{\sqrt{30}}$$

The angle $\bar{\ } = \cos^{-1}\left(\frac{5}{\sqrt{30}}\right) = 24.1°$.

The most relevant document is $\mathbf{d} = \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T$.

(d) For the given vectors $\mathbf{q} = \begin{pmatrix}0.5 & 0 & 0 & 0.5\end{pmatrix}^T$ and $\mathbf{d} = \begin{pmatrix}0 & 2 & 5 & 0\end{pmatrix}^T$ we can see that these are orthogonal $(90°)$ because $\mathbf{q}\cdot\mathbf{d} = 0$. This means that the query term does not exist in the document.

(e) The vectors $\mathbf{q} = \begin{pmatrix}0.5 & 0 & 0 & 0.5\end{pmatrix}^T$ and $\mathbf{d} = \begin{pmatrix}2 & 0 & 0 & 2\end{pmatrix}^T$ are parallel because they are multiples of each other, $\mathbf{d} = 4\mathbf{q}$. The angle between the vectors is $0°$. This means the query term is closely related to the document.

25. (a) We test the given three vectors for linearly independence. Consider the linear combination:

$$k_1\begin{pmatrix}1\\0\\2\end{pmatrix} + k_2\begin{pmatrix}1\\1\\1\end{pmatrix} + k_3\begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Need to show that the only solution is $k_1 = k_2 = k_3 = 0$ for linear independence. Augment the Right Hand Side with the column vectors on the left:

$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|c}1 & 1 & 3 & 0\\0 & 1 & 1 & 0\\2 & 1 & 5 & 0\end{array}\right)$$

Executing the row operation:

$$\begin{matrix}R_1\\R_2\\R_3^\dagger = R_3 - 2R_1\end{matrix}\left(\begin{array}{ccc|c}1 & 1 & 3 & 0\\0 & 1 & 1 & 0\\0 & -1 & -1 & 0\end{array}\right)$$

Adding the bottom two rows gives

$$\begin{matrix}R_1\\R_2\\R_3^\dagger + R_2\end{matrix}\left(\begin{array}{ccc|c}1 & 1 & 3 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right)$$

Since we have a row of zeros so the matrix consisting of the columns $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3$ are linearly *dependent* so they cannot form a basis for $\mathbb{R}^3$.

You might observe from the start that $2\mathbf{b}_1+\mathbf{b}_2=\mathbf{b}_3$.

(b) Similarly we have

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right)$$

Executing the row operation:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^\dagger = \mathrm{R}_3 - 2\mathrm{R}_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

Adding the bottom two rows:

$$\begin{matrix} \\ \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^\dagger + \mathrm{R}_2 \end{matrix} \begin{matrix} \begin{matrix} k_1 & k_2 & k_3 & \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \end{matrix} \quad \Rightarrow \quad k_1 = k_2 = k_3 = 0$$

Since the three given vectors are linearly independent so they form a basis for $\mathbb{R}^3$.

We need to write the vector $\mathbf{x}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ in terms of $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3$:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array}\right)$$

Copying the above row operations:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^\dagger = \mathrm{R}_3 - 2\mathrm{R}_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

Adding the bottom two rows:

$$\begin{matrix} \\ \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3^\dagger + \mathrm{R}_2 \end{matrix} \begin{matrix} \begin{matrix} k_1 & k_2 & k_3 & \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right) \end{matrix}$$

From the bottom row we have $k_3=-3$. Substituting this into the middle row gives

$$k_2 - 3 = 2 \ \Rightarrow\ k_2 = 5$$

Putting these two values into the top row yields:

$$k_1 + k_2 + 3k_3 = 1 \ \Rightarrow\ k_1 + 5 + 3(-3) = 1 \ \Rightarrow\ k_1 = 5$$

Hence $\mathbf{x} = 5\mathbf{b}_1 + 5\mathbf{b}_2 - 3\mathbf{b}_3$. (Check this is correct.)

26. We have

$$\mathbf{A} = (\mathbf{u} \quad \mathbf{v}) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_n & v_n \end{pmatrix} \text{ and so } \mathbf{A}^T = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

Multiplying these two matrices gives

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_n & v_n \end{pmatrix}$$

$$= \begin{pmatrix} u_1^2 + u_2^2 + \cdots + u_n^2 & u_1v_1 + u_2v_2 + \cdots + u_nv_n \\ v_1u_1 + v_2u_2 + \cdots + v_nu_n & v_1^2 + v_2^2 + \cdots + v_n^2 \end{pmatrix} = \begin{pmatrix} \|\mathbf{u}\|^2 & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \|\mathbf{v}\|^2 \end{pmatrix}$$

If **u** and **v** are orthonormal then

$$\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = 1 \text{ and } \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = 0$$

Entering these numbers into the above matrix $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \|\mathbf{u}\|^2 & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \|\mathbf{v}\|^2 \end{pmatrix}$ gives

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

27. *Proof*.

We are given $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ so by using the following result of chapter 2:

Let **u** be in $\mathbb{R}^n$ then		
(2.8)	$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$	[Positive Root]

We have

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \end{aligned}$$

From the last line we have $\mathbf{u}\cdot\mathbf{v} = 0$. By

Two vectors **u** and **v** in $\mathbb{R}^n$ are said to be **perpendicular** or **orthogonal** $\Leftrightarrow$	
(2.5)	$\mathbf{u}\cdot\mathbf{v} = 0$

Hence the vectors **u** and **v** are orthogonal.

■

28. (a) We have

$$\|\mathbf{u}\|^2 = \left\|\begin{pmatrix}32 & 16 & 8 & 4 & 2 & 1\end{pmatrix}^T\right\|$$
$$= 32^2 + 16^2 + 8^2 + 4^2 + 2^2 + 1 = 1365$$

Taking the square root gives the length

$$\|\mathbf{u}\| = \sqrt{1365} = 36.946 \text{ (3dp)}$$

(b) (i) The norm or length of $\mathbf{u}_n$ is given by

$$\|\mathbf{u}_n\|^2 = \left\|\begin{pmatrix}r^{n-1} & r^{n-2} & \cdots & r^2 & r & 1\end{pmatrix}^T\right\|^2$$
$$= \left(r^{n-1}\right)^2 + \left(r^{n-2}\right)^2 + \cdots + \left(r^2\right)^2 + r^2 + 1^2$$
$$= r^{2(n-1)} + r^{2(n-2)} + \cdots + r^4 + r^2 + 1$$

Using the formula given in the hint $1 + r + r^2 + r^3 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ with *r* being replaced by r^2 we have

$$\|\mathbf{u}_n\|^2 = r^{2(n-1)} + r^{2(n-2)} + \cdots + r^4 + r^2 + 1$$
$$= \frac{1 - r^{2n}}{1 - r^2}$$

Taking the square root gives

$$\|\mathbf{u}_n\| = \sqrt{\frac{1 - r^{2n}}{1 - r^2}}$$

(ii) We can use the formula of part (i) but with $n \to \infty$ we have $\lim_{n \to \infty}\left(r^n\right) = 0$ and so

$$\|\mathbf{u}_\infty\| = \sqrt{\frac{1}{1 - r^2}}$$

29. (a) We can write any non-unit vectors which form a basis for $\mathbb{R}^4$. A general example is:

$$\left\{\begin{pmatrix}\sim\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\ \\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\ \text{¥}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\ \text{¢}\end{pmatrix}\right\} \text{ where } \sim, \quad , \text{¥}, \text{¢} \neq 0, 1$$

(b) A set with zero vector cannot be a basis. An example is

$$\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$$

30. *Proof*.

We are given that $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent set of vectors. This means that

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O} \quad \Rightarrow \quad k_1=k_2=\cdots=k_n=0$$

Required to prove that

$$d_1(c\mathbf{v}_1)+d_2(c\mathbf{v}_2)+\cdots+d_n(c\mathbf{v}_n)=\mathbf{O} \quad \Rightarrow \quad d_1=d_2=\cdots=d_n=0$$

Considering $d_1(c\mathbf{v}_1)+d_2(c\mathbf{v}_2)+\cdots+d_n(c\mathbf{v}_n)=\mathbf{O}$ we have

$$d_1(c\mathbf{v}_1)+d_2(c\mathbf{v}_2)+\cdots+d_n(c\mathbf{v}_n)=c[d_1\mathbf{v}_1+d_2\mathbf{v}_2+\cdots+d_n\mathbf{v}_n]=\mathbf{O}$$

We are given that $c\neq 0$ therefore

$$d_1\mathbf{v}_1+d_2\mathbf{v}_2+\cdots+d_n\mathbf{v}_n=\mathbf{O} \qquad (*)$$

We are also told that $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent. By

Definition (2.19) . We say vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots$ and $\mathbf{v}_n$ in $\mathbb{R}^n$ are **linearly independent** $\Leftrightarrow$ the only real scalars $k_1,\ k_2,\ k_3,\ \cdots$ and k_n which satisfy: $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$ are $k_1=k_2=k_3=\cdots=k_n=0$

Applying this Definition (2.19) to our linear combination (*) gives

$$d_1=d_2=\cdots=d_n=0$$

Hence $\{c\mathbf{v}_1,\ c\mathbf{v}_2,\ \cdots,\ c\mathbf{v}_n\}$ is a linearly independent set of vectors.

■

31. The error occurs in the last line:

$$k_1(\mathbf{Au}_1)+k_2(\mathbf{Au}_2)+\cdots+k_n(\mathbf{Au}_n)=\mathbf{O}$$
$$\mathbf{A}(k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_n\mathbf{u}_n)=\mathbf{O}$$
$$k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_n\mathbf{u}_n=\mathbf{O} \longleftarrow$$

Error occurs here.

Why?

The matrix **A** may not be invertible, it may be a singular matrix.

32. There is no error in the given derivation.

33. We need to prove $\mathbf{Au}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{A}^T\mathbf{v}$:

Proof.

We use (2.4) on the Left Hand Side of $\mathbf{Au}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{A}^T\mathbf{v}$:

(2.4)	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}$

We have

$$\mathbf{Au}\cdot\mathbf{v}=(\mathbf{Au})^T\ \mathbf{v}=\mathbf{u}^T\mathbf{A}^T\mathbf{v}=\mathbf{u}^T\left(\mathbf{A}^T\mathbf{v}\right)=\mathbf{u}\cdot\mathbf{A}^T\mathbf{v}$$

This completes our proof.

■

34. To show that **u** is orthogonal to the given linear combination we need to prove the dot product of **u** and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is zero.

Proof.

We examine the dot product of **u** and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$:

$$\begin{aligned}\mathbf{u}\cdot\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k\right) &= \mathbf{u}\cdot\left(c_1\mathbf{v}_1\right) + \mathbf{u}\cdot\left(c_2\mathbf{v}_2\right) + \cdots + \mathbf{u}\cdot\left(c_k\mathbf{v}_k\right) && \left[\text{By (2.6)(i)} \quad (\mathbf{x}+\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}\right] \\ &= c_1\left(\mathbf{u}\cdot\mathbf{v}_1\right) + c_2\left(\mathbf{u}\cdot\mathbf{v}_2\right) + \cdots + c_k\left(\mathbf{u}\cdot\mathbf{v}_k\right) && \left[\text{By (2.6)(iii)} \quad (c\mathbf{x})\cdot\mathbf{y} = c(\mathbf{x}\cdot\mathbf{y})\right] \\ &= c_1(0) + c_2(0) + \cdots + c_k(0) && \left[\text{Because } \mathbf{u} \text{ is orthogonal to } \mathbf{v}_1, \cdots, \mathbf{v}_k\right] \\ &= 0\end{aligned}$$

Since $\mathbf{u}\cdot\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k\right) = 0$ so vector **u** is orthogonal to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

■

35. (i) Clearly

$$\left\|\mathbf{v}\right\|^2 = \left\|\left(\frac{1}{3},\ 0,\ 0,\ 0,\ \cdots\right)\right\|^2 = \frac{1}{9} + 0 + 0 + \cdots = \frac{1}{9}$$

Since 1/9 is a real number so it is a member of $\mathbb{R}^\infty$.

(ii) By using the given definition of length we have

$$\begin{aligned}\left\|\mathbf{u}\right\|^2 &= 1^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{27}\right)^2 + \left(\frac{1}{81}\right)^2 + \cdots \\ &= 1^2 + \frac{1}{9} + \left(\frac{1}{9}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{9}\right)\left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)\left(\frac{1}{9}\right)^3 + \cdots \\ &= \frac{1}{1-\frac{1}{9}} = \frac{1}{8/9} = \frac{9}{8} \qquad [\text{By Hint}]\end{aligned}$$

Since 9/8 is a real number so the given vector **u** is a member of the infinite dimensional Euclidean space.

(iii) Taking the square root gives $\left\|\mathbf{u}\right\| = \frac{3}{\sqrt{8}}$.

(iv) Using the given definition we have

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{3}(1) + 0\left(\frac{1}{3}\right) + 0\left(\frac{1}{9}\right) + 0\left(\frac{1}{27}\right) + \cdots = \frac{1}{3}$$

(v) Using the angle formula of chapter 2:

(2.12)	$\cos\left(^{-}\right) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|}$	provided **u** and **v** are non-zero vectors.

Substituting $\mathbf{u}\cdot\mathbf{v} = \frac{1}{3}$, $\left\|\mathbf{u}\right\| = \frac{3}{\sqrt{8}}$ and $\left\|\mathbf{v}\right\| = \frac{1}{3}$ into this formula gives

$$\cos\left(^{-}\right)=\frac{1/3}{\left(3/\sqrt{8}\right)\left(1/3\right)}=\frac{\sqrt{8}}{3}\quad\Rightarrow\quad {}^{-}=\cos^{-1}\left(\frac{\sqrt{8}}{3}\right)=19.47^\circ$$