Complete Solutions to Problems of Chapter 3 (Only available to tutors.)

1. (i) By applying row operations to matrix **A** we obtain $\mathbf{R}=\begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}$. There is only one-zero row so rank of matrix **A** is 1.

(ii) For the null space we solve the linear system $\mathbf{Rx}=\mathbf{O}$:

$$\mathbf{Rx}=\begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

By the top row we have

$$x+2y+3z=0 \Rightarrow x=-2y-3z$$

Let $y=s$ and $z=t$ then $x=-2s-3t$ where $s,\ t\in\mathbb{R}$. Writing our solution in vector form gives:

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2s-3t\\s\\t\end{pmatrix}=s\begin{pmatrix}-2\\1\\0\end{pmatrix}+t\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

Hence our null space N is given by

$$N=\left\{s\mathbf{u}+t\mathbf{v}\ \middle|\ \mathbf{u}=\begin{pmatrix}-2\\1\\0\end{pmatrix}\text{ and }\mathbf{v}=\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$

(iii) This is a non-homogeneous equation so by Proposition (3.35) we have

> The solution of $\mathbf{Ax}=\mathbf{b}$ consists of two parts:
>
> x=(Homogeneous Solution) + (Particular Solution)

By applying the above row operations for null space we obtain

$$\mathbf{Rx}=\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow x=1-2y-3z$$

Therefore our solution **x** is given by null space solution plus the particular solution:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2s-3t \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{Part Soln}} + \underbrace{s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{\text{Hom Soln}}$$

2. A basis *B* for M_{33} is

$$B = \left\{ \mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{E}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{E}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \mathbf{E}_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Since the basis for M_{33} consists of 9 matrices so $\dim(M_{33}) = 9$.

The linear combination of matrix **A** in terms of our basis is

$$\mathbf{A} = \mathbf{E}_1 + 2\mathbf{E}_2 + 3\mathbf{E}_3 + \cdots + 9\mathbf{E}_9$$

3. The matrix **A** has 30 rows and 25 columns. Using the following proposition:

> Proposition (3.29). Let **A** be any matrix. Then
>
> Row rank of matrix **A** = Column rank of matrix **A**

This means that the row and column rank of a matrix are equal. Hence the largest possible rank can only be 25 because we have 25 columns.

4. By applying row operations we have

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{matrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Rightarrow \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2{*} = \mathrm{R}_2 - 3\mathrm{R}_1 \\ \mathrm{R}_3{*} = \mathrm{R}_3 - 5\mathrm{R}_1 \end{matrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2{*} \\ \mathrm{R}_3{**} = \mathrm{R}_3{*} - 2\mathrm{R}_2{*} \end{matrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2{*}/(-2) \\ \mathrm{R}_3{**} \end{matrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The row space is given by the space spanned by the non-zero rows in the last matrix which is

$$\text{Row Space } = \left\{ s\mathbf{u}+t\mathbf{v} \;\middle|\; \mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}0\\1\end{pmatrix} \right\}$$

Since the vectors **u** and **v** are linearly independent so the row space is the whole of $\mathbb{R}^2$.

5. (a) (i) Writing the augmented matrix for the given linear system we have

$$\begin{matrix}R_1\\R_2\end{matrix}\left(\begin{array}{ccc|c}2&1&1&0\\1&-1&1&0\end{array}\right)$$

Carrying out the row operation $2R_2-R_1$ gives

$$\begin{matrix}\\R_1\\2R_2-R_1\end{matrix}\begin{matrix}\begin{matrix}x&y&z\end{matrix}\\\left(\begin{array}{ccc|c}2&1&1&0\\0&-3&1&0\end{array}\right)\end{matrix}$$

From the bottom row we have $-3y+z=0 \Rightarrow z=3y$. Let $y=s$ where s is any real number then $z=3s$. From the top row we have

$$2x+y+z=0 \Rightarrow x=-\frac{y+z}{2}=-\frac{3s+s}{2}=-2s$$

Our homogeneous solution is

$$\mathbf{x}_H=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2s\\s\\3s\end{pmatrix}=s\begin{pmatrix}-2\\1\\3\end{pmatrix}$$

(ii) We have inhomogeneous equation so the general solution is given by

x=(Homogeneous Solution) + (Particular Solution)

We found the homogeneous solution in part (i).
The particular solution is given by:

$$\begin{matrix}R_1\\R_2\end{matrix}\left(\begin{array}{ccc|c}2&1&1&4\\1&-1&1&5\end{array}\right)$$

$$\begin{matrix}\\R_1\\2R_2-R_1\end{matrix}\begin{matrix}\begin{matrix}x&y&z\end{matrix}\\\left(\begin{array}{ccc|c}2&1&1&4\\0&-3&1&6\end{array}\right)\end{matrix}$$

Opening up the bottom row gives

$$-3y+z=6 \quad\Leftrightarrow\quad z=6+3y=6+3s$$

Substituting this into the top row yields

$$2x+y+z=4 \quad\Leftrightarrow\quad x=2-\frac{y+z}{2}=2-\frac{s+6+3s}{2}=2-2s-3=-1-2s$$

The general solution in vector form is given by

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1-2s\\s\\6+3s\end{pmatrix}=\underbrace{\begin{pmatrix}-1\\0\\6\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$$

(iii) Similar to part (ii):

$$\begin{matrix}R_1\\R_2\end{matrix}\left(\begin{array}{ccc|c}2&1&1&4\\1&-1&1&1\end{array}\right)$$

$$\begin{matrix}&x&y&z\end{matrix}$$

$$\begin{matrix}R_1\\2R_2-R_1\end{matrix}\left(\begin{array}{ccc|c}2&1&1&4\\0&-3&1&-2\end{array}\right)$$

By the bottom row we have

$$z=-2+3s$$

Substituting this into the top row yields

$$x=2-\frac{y+z}{2}=2-\frac{s-2+3s}{2}=2-2s+1=3-2s$$

The general solution is given by

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3-2s\\s\\-2+3s\end{pmatrix}=\underbrace{\begin{pmatrix}3\\0\\-2\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$$

(iv) Since $\mathbf{b}_3=2\mathbf{b}_1-\mathbf{b}_2$ so we can adjust the particular solutions of parts (ii) and (iii).

Let $\mathbf{p}'=\begin{pmatrix}-1\\0\\6\end{pmatrix}$ be the particular solution for part (ii) and $\mathbf{p}''=\begin{pmatrix}3\\0\\-2\end{pmatrix}$ be the particular solution for part (iii). Then the particular solution for $\mathbf{b}_3$ is given by

$$2\mathbf{p}'-\mathbf{p}''=2\begin{pmatrix}-1\\0\\6\end{pmatrix}-\begin{pmatrix}3\\0\\-2\end{pmatrix}=\begin{pmatrix}-2-3\\0\\12+2\end{pmatrix}=\begin{pmatrix}-5\\0\\14\end{pmatrix}$$

Hence the general solution is given by

$$\mathbf{x}=\underbrace{\begin{pmatrix}-5\\0\\14\end{pmatrix}}_{\text{Particular soln}}+\underbrace{s\begin{pmatrix}-2\\1\\3\end{pmatrix}}_{\text{Homogeneous soln}}$$

(b) *Proof*.

We have $\mathbf{Ap}_1=\mathbf{b}_1$ and $\mathbf{Ap}_2=\mathbf{b}_2$ because these are the particular solutions for $\mathbf{b}_1$ and $\mathbf{b}_2$. Substituting for $\mathbf{b}_1$ and $\mathbf{b}_2$ into $\mathbf{Ax}=k\mathbf{b}_1+c\mathbf{b}_2$ gives:

$$\mathbf{A}(k\mathbf{p}_1+c\mathbf{p}_2)=k(\mathbf{Ap}_1)+c(\mathbf{Ap}_2)=k\mathbf{b}_1+c\mathbf{b}_2$$

Since $\mathbf{A}\left(k\mathbf{p}_1 + c\mathbf{p}_2\right) = k\mathbf{b}_1 + c\mathbf{b}_2$ so the particular solution is given by $k\mathbf{p}_1 + c\mathbf{p}_2$. By Proposition (3.35) we have

The solution of $\mathbf{Ax} = \mathbf{b}$ consists of two parts:

x=(Homogeneous Solution) + (Particular Solution)

We have $\mathbf{x} = \underbrace{\mathbf{x}_H}_{\text{Homogeneous soln}} + k\mathbf{p}_1 + c\mathbf{p}_2$. This completes our proof.

■

6. In each case we use the result of the previous question part (b), $\mathbf{x} = \mathbf{x}_H + k\mathbf{p}_1 + c\mathbf{p}_2$.

(a) $\mathbf{x} = \mathbf{x}_H - 6\mathbf{p}_1$

(b) $\mathbf{x} = \mathbf{x}_H + 3\mathbf{p}_2$

© $\mathbf{x} = \mathbf{x}_H + 5\mathbf{p}_1 - 73\mathbf{p}_2$

7. The given matrices are $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}$, $\begin{pmatrix} -4 & -11 \\ 9 & -1 \end{pmatrix}$.

(a) Let us check if the 3 given matrices are linearly independent:

$$k_1\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + k_2\begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix} + k_3\begin{pmatrix} -4 & -11 \\ 9 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating entries gives the equations

$$\begin{aligned} k_1 + 2k_2 - 4k_3 &= 0 \\ 2k_1 + 5k_2 - 11k_3 &= 0 \\ 3k_1 - k_2 + 9k_3 &= 0 \\ 4k_1 + 3k_2 - k_3 &= 0 \end{aligned}$$

Writing this as an augmented matrix and carrying out row operations we have

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 5 & -11 & 0 \\ 3 & -1 & 9 & 0 \\ 4 & 3 & -1 & 0 \end{array}\right)$$

Executing row operations:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2^* = \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3^* = \mathrm{R}_3 - 3\mathrm{R}_1 \\ \mathrm{R}_4^* = \mathrm{R}_4 - 4\mathrm{R}_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -7 & 21 & 0 \\ 0 & -5 & 15 & 0 \end{array}\right)$$

Carrying out the following row operations:

$$\begin{array}{r} \\ R_1 \\ R_2* \\ R_3*+7R_2* \\ R_4*+5R_2* \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

From the second row we have

$$k_2 = 3k_3$$

From the top row we have

$$k_1 + 2k_2 - 4k_3 = 0$$

Substituting $k_2 = 3k_3$ into the last equation gives

$$k_1 + 2(3k_3) - 4k_3 = 0 \quad \Rightarrow \quad k_1 = -2k_3$$

Let $k_3 = 1$ then $k_2 = 3k_3 = 3(1) = 3$ and $k_1 = -2k_3 = -2(1) = -2$. We have

$$-2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 3\begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} -4 & -11 \\ 9 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence the given matrices are linearly dependent.

(b) In part(a) of the above the row echelon form has 2 non-zero rows so the dimension of the given subspace is 2.

8. Using row operations on the given matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

Executing the following row operations:

$$\begin{array}{r} R_1 \\ R_2^* = R_2 - 5R_1 \\ R_3^* = R_3 - 9R_1 \\ R_4^* = R_4 - 13R_1 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{pmatrix}$$

Note that the third row is double the second row and bottom row is 3 times the second row. Carrying out the appropriate row operations we obtain:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space are the non-zero rows:

$$B_r = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

A basis for the column space is the columns with leading ones:

$$B_c = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The rank of matrix **A** is 2.

Solving the homogeneous equation $\mathbf{Ax} = \mathbf{O}$:

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the second row we have

$$y + 2z + 3w = 0 \quad \Rightarrow \quad y = -2z - 3w$$

Let $z = s$ and $w = t$ then $y = -2s - 3t$. Using the top row we have

$$x + 2y + 3z + 4w = 0 \;\Rightarrow\; x = -2y - 3z - 4w$$

Substituting $z = s,\ w = t$ and $y = -2s - 3t$ into this $x = -2y - 3z - 4w$ gives

$$\begin{aligned} x &= -2(-2s - 3t) - 3s - 4t \\ &= 4s + 6t - 3s - 4t \\ &= s + 2t \end{aligned}$$

Our solution set is given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

A basis for the solution space is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

9. The set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ is linearly dependent $\Leftrightarrow$ there are non-zero scalars $k_1,\ k_2,\ \cdots,\ k_n$ such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{O}$$

10. Let *k* and *c* be scalars such that

$$k\mathbf{u} + c\mathbf{v} = k\begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0.009\\0.001\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

From the bottom row we have $c = 1000$. Substituting this into the top row yields

$$k + 9 = 1 \quad \Rightarrow \quad k = -8$$

We have $-8\mathbf{u} + 1000\mathbf{v} = \mathbf{w}$. Therefore the coordinates in terms of the basis **u** and **v** is $\begin{pmatrix}-8\\1000\end{pmatrix}$.

11. (i) Since $\mathbf{u} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}1\\10^{-6}\end{pmatrix}$ are not multiples of each other so they are linearly independent. Hence they form a basis for $\mathbb{R}^2$. Note that they are very close to being dependent because they are nearly in the same direction.

(ii) We need to write $\mathbf{w} = \begin{pmatrix}1\\1\end{pmatrix}$ as a linear combination of **u** and **v**:

$$k\begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}1\\10^{-6}\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad \Rightarrow \quad \begin{matrix}k + c = 1\\ 10^{-6}c = 1\end{matrix} \quad \Rightarrow \quad c = 10^6,\; k = 1 - 10^6 = -999\,999$$

We have

$$1\,000\,000\begin{pmatrix}1\\10^{-6}\end{pmatrix} - 999\,999\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

12. No matrices of the given format do *not* form a subspace of M_{25} because if

$$\mathbf{B} = \begin{pmatrix}b_{11} & b_{12} & 0 & 1 & b_{15}\\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25}\end{pmatrix}$$

Then $\mathbf{A} + \mathbf{B}$ is not in *S* because the entry in the first row and fourth column will be 2 not 1. Therefore *S* is not a subspace of M_{25}.

13. Consider the vector space M_{22}. Let $\mathbf{A} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}-1 & -2\\3 & 4\end{pmatrix}$ then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} + \begin{pmatrix}-1 & -2\\3 & 4\end{pmatrix} = \begin{pmatrix}0 & 0\\6 & 8\end{pmatrix}$$

The matrix $\mathbf{A} + \mathbf{B} = \begin{pmatrix}0 & 0\\6 & 8\end{pmatrix}$ is *not* invertible so is not a member of the set *S*. By the following proposition of chapter 3:

Proposition (3.7). A non - empty subset *S* containing vectors **u** and **v** is a subspace of a vector space *V* $\Leftrightarrow$ any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in *S* (k and c are scalars).

We conclude that *S* is *not* a subspace of M_{22} because the linear combination $\mathbf{A}+\mathbf{B}$ is *not* in *S*.

14. Let *f* and *g* be differentiable functions in *S*. Consider the linear combination

$kf + cg$ where *k*, *c* are scalars

The derivative of this is given by

$$(kf + cg)' = kf' + cg'$$

Since *f* and *g* are differentiable so $kf + cg$ is differentiable which means the linear combination is in *S*. Hence the set *S* of differentiable functions is a subspace of *V*.

15. We are given $x - 2y - 3z = 0$ so the solution of this is

$$x - 2y - 3z = 0 \quad \Rightarrow \quad x = 2y + 3z$$

Let $y = t$ and $z = s$ then $x = 2t + 3s$. We have

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t+3s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = t\mathbf{u} + s\mathbf{v} \quad \text{where} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Since the vectors **u** and **v** are linearly independent because they are not multiples of each other so they form a basis for the given plane.

16. Labelling the rows of the given matrix and apply row operations:

$$\begin{matrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \end{matrix} \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -6 & 1 & 0 \end{pmatrix}$$

Interchanging top and bottom row gives

$$\begin{matrix} \mathrm{R}_1{*} = \mathrm{R}_4 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4{*} = \mathrm{R}_1 \end{matrix} \begin{pmatrix} 1 & -6 & 1 & 0 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & 2 & -1 \end{pmatrix}$$

Carrying out the following row operations:

$$\begin{matrix} \mathrm{R}_1{*} \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4{**} = 2\mathrm{R}_4{*} + 3\mathrm{R}_2{*} \end{matrix} \begin{pmatrix} 1 & -6 & 1 & 0 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 7 \end{pmatrix}$$

Executing

$$\begin{matrix} R_1{}^* = R_4 \\ R_2 \\ R_3 \\ R_4{}^{**} - 7R_3 \end{matrix} \begin{pmatrix} 1 & -6 & 1 & 0 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By examining the entries in this matrix we will not get any more zero rows so the rank of the matrix is 3 because we have 3 non-zero rows. Hence $rank(\mathbf{A}) = 3$.

Applying:

Theorem (3.34). The Dimension Theorem of Matrices.

If **A** is a matrix with *n* columns (number of unknowns) then $nullity(\mathbf{A}) + rank(\mathbf{A}) = n$.

We have $nullity(\mathbf{A}) = n - rank(\mathbf{A}) = 4 - 3 = 1$. This means the null space is of dimension 1.

Let us find the null space. Using the above derived matrix:

$$\begin{matrix} x & y & z & w & \end{matrix}$$
$$\left(\begin{array}{cccc|c} 1 & -6 & 1 & 0 & 0 \\ -2 & 4 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the third row we have

$z + w = 0$ where *w* is our free variable

Let $w = r$ where *r* is any real number. Then $z = -w = -r$. Substituting these into the second row:

$$-2x + 4y + z + 3w = 0 \;\Rightarrow\; -2x + 4y - r + 3r = 0 \;\Rightarrow\; -2x + 4y = -2r$$

From the first row we have

$$x - 6y + z = 0 \;\Rightarrow\; x - 6y = r$$

Solving these two equations $-2x + 4y = -2r$ and $x - 6y = r$ gives

$$x = r,\; y = 0$$

Our solution is $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ -r \\ r \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. The null space is given by $N(\mathbf{A}) = \left\{ r\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.

We need to solve the non-homogeneous linear system. We use the following:

Proposition (3.35). Let $\mathbf{x}_P$ be a particular solution of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x}_H$ be the solution to the homogeneous system $\mathbf{Ax} = \mathbf{O}$. All the solutions of $\mathbf{Ax} = \mathbf{b}$ are of the form $\mathbf{x}_P + \mathbf{x}_H$.

The homogeneous solution is the null space solution, that is $\mathbf{x}_H = r\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$. We need to find

$\mathbf{x}_P$ such that $\mathbf{Ax}_p = \mathbf{b}$:

$$\begin{pmatrix}3 & -6 & 2 & -1\\ -2 & 4 & 1 & 3\\ 0 & 0 & 1 & 1\\ 1 & -6 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-1\\3\\1\\0\end{pmatrix}$$

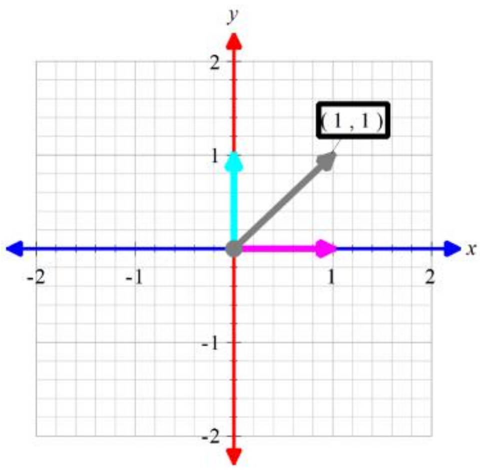
We can write this as a linear combination:

$$x\begin{pmatrix}3\\-2\\0\\1\end{pmatrix}+y\begin{pmatrix}-6\\4\\0\\-6\end{pmatrix}+z\begin{pmatrix}2\\1\\1\\1\end{pmatrix}+w\begin{pmatrix}-1\\3\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\3\\1\\0\end{pmatrix}$$

Note that the last column vector of matrix **A** is the vector **b**. Therefore the particular solution is $x = y = z = 0,\ w = 1$. Hence our general solution by the above Proposition (3.35):

$$\mathbf{x}_P + \mathbf{x}_H = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} + r\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

17. No because if $\mathbf{u} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}0\\1\end{pmatrix}$ then $\mathbf{u}+\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ which is not on any of axes:



18. Yes. Let $S = \{y = \sqrt{2}x \mid x,\ y \in \mathbb{R}\}$ and **u** and **v** be members of this set. Then

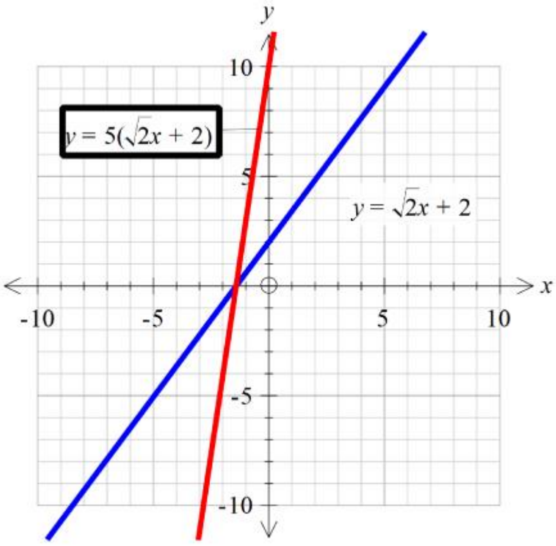
$\mathbf{u}+\mathbf{v}$ is in the set *S*

$k\mathbf{u}$ is also in *S*

Since the set is closed under vector addition and scalar multiplication so *S* is a vector space.

19. No because if we let $S = \{ y = \sqrt{2}x + 2 \mid x,\ y \in \mathbb{R} \}$ and **u** be a member of this set then

$$5\mathbf{u} = 5\sqrt{2}x + 2\sqrt{5} \text{ is not a member of } S$$



$5\mathbf{u} = 5\sqrt{2}x + 2\sqrt{5}$ does not lie on the line $y = \sqrt{2}x + 2$.

20. Writing the given vectors in a matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 2 & -1 & 0 & 1 \\ 6 & 1 & 4 & -5 \\ 28 & -2 & 12 & -10 \end{pmatrix}$$

Using the following row operations:

$$\begin{matrix} R_1 \\ R_2{*} = R_2 - 3R_1 \\ R_3{*} = R_3 - 14R_1 \end{matrix} \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 4 & 4 & -8 \\ 0 & 12 & 12 & -24 \end{pmatrix}$$

Executing

$$\begin{matrix} R_1 \\ R_2{*}/4 \\ R_3{*} - 3R_2{*} \end{matrix} \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are 2 non-zero rows in row echelon form so the three given vectors span a subspace of $\mathbb{R}^4$ which is of dimension 2. A basis for this space are the first two row vectors of the above matrix:

$$\left\{\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-2\end{pmatrix}\right\}$$

21. *Proof.*

Let $\mathbf{A}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x_1 & x_2 & \cdots & x_n\end{pmatrix}^T$ then

$$\mathbf{Ax}=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

Writing out the linear combination and using the given condition that the column vectors are linearly independent we have

$$\mathbf{Ax}=x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots x_n\mathbf{v}_n=\mathbf{O} \quad \Rightarrow \quad x_1=x_2=\cdots=x_n=0$$

Hence $\mathbf{x}=\mathbf{O}$.

■

22. (a) Writing the given vectors in a matrix and labelling the rows gives:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\1 & -3 & 8 & -7 & 1 & 2\\5 & -5 & 30 & -13 & 13 & 18\end{pmatrix}$$

Executing the row operations $\mathrm{R}_2-\mathrm{R}_1$ and $\mathrm{R}_3-5\mathrm{R}_1$ yields:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{\dagger}=\mathrm{R}_2-\mathrm{R}_1\\\mathrm{R}_3^{\dagger}=\mathrm{R}_3-5\mathrm{R}_1\end{matrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\0 & -5 & 5 & -11 & -4 & -4\\0 & -15 & 15 & -33 & -12 & -12\end{pmatrix}$$

Carrying out the row operation $\mathrm{R}_3^{\dagger}-3\mathrm{R}_2^{\dagger}$:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{\dagger}\\\mathrm{R}_3^{\dagger}-3\mathrm{R}_2^{\dagger}\end{matrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\0 & -5 & 5 & -11 & -4 & -4\\0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

The top two row vectors of this matrix are linearly independent so they form a basis for the subspace of $\mathbb{R}^6$. Hence a basis for this subspace is

$$\left\{\mathbf{u},\ \begin{pmatrix}0 & -5 & 5 & -11 & -4 & -4\end{pmatrix}^T\right\}$$

The dimension of this space is 2 as we only have 2 vectors in the basis.

(b) Similarly we have

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 2 & 1 & 3 & -1 & 4 & -1 \\ 1 & -1 & 2 & -2 & 3 & -3 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix}$$

Carrying out the following row operations:

$$\begin{matrix} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_2 \end{matrix} \begin{pmatrix} 2 & 1 & 3 & -1 & 4 & -1 \\ 0 & -3 & -4 & 0 & -5 & -1 \\ 0 & 10 & -5 & 15 & -10 & 20 \end{pmatrix}$$

Multiplying the middle row by -1 and bottom row by 1/5 gives

$$\begin{pmatrix} 2 & 1 & 3 & -1 & 4 & -1 \\ 0 & 3 & 4 & 0 & 5 & 1 \\ 0 & 2 & -1 & 3 & -2 & 4 \end{pmatrix}$$

Whichever row operation we carry out from here we will not result in a row of zeros so the given vectors are linearly independent. These vectors span a subspace of $\mathbb{R}^6$ and because they are linearly independent so they form a basis for this subspace. Hence a basis (axes) is $\left\{\mathbf{u},\ \begin{pmatrix}0 & 3 & 4 & 0 & 5 & 1\end{pmatrix}^T,\ \begin{pmatrix}0 & 2 & -1 & 3 & -2 & 4\end{pmatrix}^T\right\}$. The dimension of this subspace is 3 because we have 3 vectors in the basis. (Actually the 3 given vectors are linearly independent and also form a basis for this subspace.)

23. The dimension of P_2 is 3, so if we can show the given set of 3 vectors are linearly independent then they form a basis for P_2. However note that

$$1 - x^2 = (1 - x)(1 + x)$$

This means that the first 2 vectors in the given set $\{1 - x,\ 1 - x^2,\ 1 + x + x^2\}$ are linearly dependent because they are multiples of each other. Hence the given set is not a basis for P_2.

24. (i) Writing the given vectors in a matrix and labelling the rows:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 10 \\ 1 & -4 & -7 \end{pmatrix} = \mathbf{A}$$

Executing the following row operations:

$$\begin{matrix} R_1 \\ R_2 + 3R_1 \\ R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 7 & 7 \\ 0 & -6 & -6 \end{pmatrix}$$

Carrying out one more row operation we have

$$\begin{matrix} R_1 \\ R_2+3R_1 \\ R_3-R_1 \end{matrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

We have a row of zeros so the given vectors are linearly dependent.

(ii) Remember the dimension of P_2 is 3. Just need to check that the given vectors are linearly independent. Consider the linear combination of the given vectors and equating to the zero vector:

$$k_1\left(x^2-3x+1\right)+k_2\left(2x^2+x-4\right)+k_3\left(-x^2+10x-7\right)=\mathbf{O}$$

(*k*'s are scalars.). Equating coefficients gives

x^2; $\quad k_1+2k_2-k_3=0$

x; $\quad -3k_1+k_2+10k_3=0$

Const; $\quad k_1-4k_2-7k_3=0$

If we write this in matrix form we end up with the above matrix **A**. This means that the given vectors are linearly dependent so they cannot form a basis for P_2.

25. (i) Writing the given equations in an augmented matrix gives

$$\begin{matrix} R_1 \\ R_2 \end{matrix}\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & -5 & 1 & 0 \end{array}\right)$$

Carrying out the row operation R_2+R_1:

$$\begin{matrix} R_1 \\ R_2+R_1 \end{matrix}\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 4 & -4 & 0 & 0 \end{array}\right)$$

Multiplying the bottom row by ¼:

$$\begin{matrix} x & y & z & \end{matrix}$$
$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

We have $x-y=0 \Rightarrow x=y$. Let $y=r$ where *r* is any real number. Then $x=r$. By expanding the top row we have

$$3x+y-z=0 \Rightarrow z=3x+y=4r$$

Our solution is $\mathbf{x}=\begin{pmatrix} r \\ r \\ 4r \end{pmatrix}=r\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$. Hence the subspace *S* is spanned by $\left\{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}\right\}$ which is a basis for *S*. This subspace *S* is of dimension 1.

(ii) We are given

$$\begin{matrix} 3x & + & y & - & z & = & 0 \\ x & - & 5y & + & z & = & 0 \end{matrix}$$ which can be written as $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$ and $\begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$

By part (i) we have a solution with $r=1$ as $\mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$.

Hence the subspace $S^{\perp}$ is spanned by the vectors $\left\{\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}\right\}$. This subspace is of dimension 2.

Note that $\dim(S)+\dim(S^{\perp})=1+2=3=\dim(\mathbb{R}^3)$.

26. First we find a basis for *S*.

Writing the given vectors in a matrix

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & -1 & 2 & -3 \\ 1 & 1 & 2 & 0 \\ 3 & -1 & 6 & -6 \end{pmatrix}$$

Executing the following row operations

$$\begin{matrix} R_1 \\ R_2-R_1 \\ R_3-3R_1 \end{matrix}\begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

Subtracting the bottom two rows gives

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for *S* is $\left\{\begin{pmatrix}1 & -1 & 2 & -3\end{pmatrix}^T, \begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T\right\}$.

Now we find a basis for *T*.

The given vectors which span the subspace *T*

$$\left\{\begin{pmatrix}0 & -2 & 0 & -3\end{pmatrix}^T, \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\right\}$$

These vectors are *not* multiples of each other so they are linearly *independent* because:

Proposition (3.12). Two vectors **u** and **v** in a vector space *V* are linearly **dependent** $\Leftrightarrow$ one of the vectors is a multiple of the other.

They form a basis for the subspace *T*.

(ii) The only basis vector which lies in both subspaces *S* and *T* is

$$\left\{\begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T\right\}$$

This is a basis for $S\cap T$.

(iii) The subspace $S+T$ is spanned by the 5 given vectors. However only the following 3 vectors are linearly independent.

$$\left\{\begin{pmatrix}1 & -1 & 2 & -3\end{pmatrix}^T,\ \begin{pmatrix}0 & 2 & 0 & 3\end{pmatrix}^T,\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T\right\}$$

Hence they form a basis for $S+T$.

27. *Proof*.

Suppose the set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\}$ is linearly dependent. Then there are scalars $k_1,\ k_2,\ \cdots,\ k_n,\ k_{n+1}$ such that

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n+k_{n+1}\mathbf{u}=\mathbf{O} \qquad (*)$$

where all the *scalars* are *not* zero. In this case $k_{n+1}\neq 0$. *Why?*

Because we are given that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ is linearly independent so

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n=\mathbf{O} \quad\Rightarrow\quad k_1=k_2=\cdots=k_n=0$$

Making **u** the subject of the formula of (*):

$$\mathbf{u}=-\frac{k_1}{k_{n+1}}\mathbf{v}_1-\frac{k_2}{k_{n+1}}\mathbf{v}_2-\cdots-\frac{k_n}{k_{n+1}}\mathbf{v}_n$$

Since the vector **u** can be written as a linear combination of the vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$ it must be in $S=span\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\}$. However we are given that **u** is *not* in *S*. We have contradiction, so our supposition $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\}$ is linearly dependent must be wrong. Hence $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n,\ \mathbf{u}\}$ is linearly *independent*.

■

28. Clearly these vectors $S=\{e^{3x},\ e^{-3x}\}$ span V because

$$\left(e^{-3x}\right)''=9e^{-3x} \quad\text{and}\quad \left(e^{3x}\right)''=9e^{-3x}$$

Both of the elements in $S=\{e^{3x},\ e^{-3x}\}$ satisfy $f''-9f=0$. To show that the set *S* is a basis for V we need them to be linearly independent.

How do we show the vectors in $\{e^{3x},\ e^{-3x}\}$ are linearly independent?

By considering the linear combination

$$k_1e^{3x}+k_2e^{-3x}=\mathbf{O} \text{ and showing } k_1=k_2=0$$

Substituting $x=0$ into this gives

$$k_1+k_2=0 \quad \text{implies } k_1=-k_2$$

Substituting $x=1/3$ into this gives

$$k_1e+k_2e^{-1}=0$$

Substituting $k_1=-k_2$ into this result yields

$$-k_2e+k_2e^{-1}=k_2\left(e+\frac{1}{e}\right)=0 \;\Rightarrow\; k_2=0$$

From $k_1 = -k_2$ we have $k_1 = 0$. We have $k_1 = k_2 = 0$ so the vectors in $\{e^{3x},\ e^{-3x}\}$ are linearly independent.
Hence $S = \{e^{3x},\ e^{-3x}\}$ is a basis for *V*.

29. The given vectors $\{f_1(x) = \sin(2x),\ f_2(x) = \cos(2x)\}$ span *V* because

$$f_1''(x) = (\sin(2x))'' = -4\sin(2x) = -4f_1(x)$$

$$f_2''(x) = (\cos(2x))'' = -4\cos(2x) = -4f_2(x)$$

They satisfy the given differential equation $f''(x) + 4f(x) = 0$.
To show that the set *S* is a basis for *V* we need them to be linearly independent.
Consider the linear combination

$$k\sin(2x) + c\cos(2x) = \mathbf{O}$$

Substituting $x = 0$ into this gives

$$c = 0$$

Substituting $x = ®/4$ into this gives

$$k\sin\left(\frac{®}{2}\right) + c\cos\left(\frac{®}{2}\right) = 0 \quad \Rightarrow \quad k = 0$$

Since $k = c = 0$ so the vectors in $\{f_1(x) = \sin(2x),\ f_2(x) = \cos(2x)\}$ are linearly independent. Hence this set of vectors is a basis for the solution space of the given differential equation.

30. (a) There are 7 entries and each one is different so

$$d(\mathbf{u},\ \mathbf{v}) = 7$$

(b) How many of the entries are different between the vectors **u** and **v** where

$$\mathbf{u} = (1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1)^T,\ \mathbf{v} = (1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1)^T\ ?$$

Only the third and fourth entries are different so

$$d(\mathbf{u},\ \mathbf{v}) = 2$$

31. *Proof.*
Consider the linear combination:

$$k_1\mathbf{u} + k_2\mathbf{A}\mathbf{u} + k_3\mathbf{A}^2\mathbf{u} + \cdots + k_m\mathbf{A}^{m-1}\mathbf{u} = \mathbf{O} \qquad (*)$$

where *k*'s are scalars. Required to prove all the scalars are zero.
Multiply both sides of (*) by $\mathbf{A}^{m-1}$:

$$k_1\mathbf{A}^{m-1}\mathbf{u} + k_2\mathbf{A}^m\mathbf{u} + k_3\mathbf{A}^{m+1}\mathbf{u} + \cdots + k_m\mathbf{A}^{2m-2}\mathbf{u} = \mathbf{O}$$

Apart from the first term on the Left Hand Side the remaining terms are all zero. *Why?*
Because $\mathbf{A}^m = \mathbf{O}$ and any matrix times the zero matrix also gives the zero matrix.

We have $k_1\mathbf{A}^{m-1}\mathbf{u}=\mathbf{O}$ but we are given that $\mathbf{A}^{m-1}\mathbf{u}\neq\mathbf{O}$ which means we must have $k_1=0$. Substituting this into (*) gives

$$k_2\mathbf{A}\mathbf{u}+k_3\mathbf{A}^2\mathbf{u}+\cdots+k_m\mathbf{A}^{m-1}\mathbf{u}=\mathbf{O} \qquad (**)$$

Multiplying both sides of this by $\mathbf{A}^{m-2}$

$$k_2\mathbf{A}^{m-1}\mathbf{u}+k_3\mathbf{A}^m\mathbf{u}+\cdots+k_m\mathbf{A}^{2m-3}\mathbf{u}=\mathbf{O}$$

By the above argument we have $k_2=0$. We can repeat this process to get

$$k_3=k_4=k_5=\cdots=k_m=0.$$

Hence the given set of vectors $\{\mathbf{u},\ \mathbf{A}\mathbf{u},\ \mathbf{A}^2\mathbf{u},\ \cdots,\ \mathbf{A}^{m-1}\mathbf{u}\}$ are linearly independent which completes our proof.

■

32. Remember $P_3,\ M_{22},\ \mathbb{R}^4$ are vector spaces of polynomials of degree 3 or less, matrices of size 2 by 2 and 4-space respectively. Of course there are an infinite number of basis for each of these vector spaces but the standard basis are

$\{1,\ x,\ x^2,\ x^3\}$ for P_3, $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ for M_{22} and

$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ for $\mathbb{R}^4$

33. (a) All the rows (or columns) of the matrix are all multiples of each other or we can say they are linearly *dependent*. The row or column space is of dimension 1.

An example of rank 1 is $\mathbf{A}=\begin{pmatrix}1&2&3\\3&6&9\\4&8&12\end{pmatrix}$. Note that bottom two rows are multiples of the top row.

(b) None of the rows or columns can be produced by a linear combination of the others. All the row (and column) vectors are linearly independent.

One of the simplest example is the identity matrix $\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ because each of the rows or columns are linearly independent. Another example is

$$\mathbf{A}=\begin{pmatrix}1&2&3\\4&5&6\\7&7&8\end{pmatrix}$$

34. Clearly as V is n dimensional so it can at most have n linearly independent vectors and the minimum number of vectors needed to span V is also n. This is because of:

> Theorem (3.22). Let V be a finite n-dimensional vector space. We have the following:
>
> (a) Any linearly *independent* set of n vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots\ ,\ \mathbf{v}_n\}$ form a basis for V.
>
> (b) Any *spanning* set of n vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \cdots\ ,\ \mathbf{u}_n\}$ form a basis for V.

35. We need to prove:

If **A** is an m by n matrix such that $n > m$ then the column vectors of matrix **A** are linearly dependent.

Proof.

Suppose the n columns of matrix **A** are linearly independent then by:

> Proposition (3.29). Let **A** be any matrix. Then
>
> Row rank of matrix **A** = Column rank of matrix **A**

The rank of matrix **A** is greater or equal to n, $rank(\mathbf{A}) \geq n$, because we have n independent columns. Remember the rank measures the number of linear independent columns (or rows) in a matrix because of question 17 of Exercise 3(e):

> Let **A** be any matrix whose rows are given by the set of linear independent vectors $S = \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \cdots,\ \mathbf{r}_n\}$. Prove that $rank(\mathbf{A}) = n$.

However there are only m rows in matrix **A** so $rank(\mathbf{A}) \leq m$. We have

$$rank(\mathbf{A}) \leq m \text{ and } rank(\mathbf{A}) \geq n \text{ where } n > m$$

This is impossible, so the columns of matrix **A** are linearly dependent.

■

36. Required to prove that if **A** is a non-square matrix then the rows or columns of matrix **A** are linearly dependent.

Proof.

Without loss of generality assume there are more columns than rows of matrix **A**. By the result of the previous question we have the columns of matrix **A** are linearly dependent. Similarly we can present the argument with more rows than columns.

■

37. We need to show that $S=\{2,\ 2x-1,\ x^2+1\}$ is a basis for P_2. By chapter 3 we have

$$\dim(P_2)=3$$

Since the number of vectors in the given set *S* is 3 so we only need to check linear independence because

> Theorem (3.22). Let *V* be a finite *n*-dimensional vector space. We have the following:
>
> (a) Any linearly *independent* set of *n* vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \cdots,\ \mathbf{v}_n\}$ form a basis for *V*.

Let $k_1,\ k_2$ and k_3 be scalars and consider the linear combination:

$$k_1(2)+k_2(2x-1)+k_3(x^2+1)=0$$
$$k_3x^2+2k_2x+(2k_1-k_2+k_3)=0$$

Equating coefficients gives $k_1=k_2=k_3=0$ so $S=\{2,\ 2x-1,\ x^2+1\}$ is linearly independent and by Theorem (3.22) we conclude that this set is a basis for P_2.

We also need to write the given vector $\mathbf{p}=x^2+1$ in terms of this basis:

$$k_3x^2+2k_2x+(2k_1-k_2+k_3)=\mathbf{p}=x^2+1$$

Again by equating coefficients we have

$$k_3=1,\ k_2=0\ \text{ and }\ 2k_1-k_2+k_3=2k_1-0+1=1\ \Rightarrow\ k_1=0$$

Hence $\mathbf{p}=0+0+1(x^2+1)$ or we can write this in terms of coordinates with respect to the given basis *S* as $\mathbf{p}_S=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

38. (i) (a) $\|\mathbf{I}\|_F=\sqrt{1^2+0^2+0^2+1^2}=\sqrt{2}$

(b) Squaring each of the entries and taking the square root of the sum:

$$\|\mathbf{A}\|_F=\sqrt{1^2+2^2+3^2+4^2}=\sqrt{30}$$

(c) Similarly we have

$$\|\mathbf{B}\|_F=\sqrt{1^2+2^2+3^2+2^2\cdots+9^2}=14$$

(d) We have

$$\|\mathbf{C}\|_F=\sqrt{1^2+2^2+\cdots+15^2}=\sqrt{770}$$

(ii) Remember the trace of a matrix is the sum of the leading diagonal terms.

(a) We have

$$\mathbf{I}^T\mathbf{I}=\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\sqrt{trace(\mathbf{I}^T\mathbf{I})}=\sqrt{1+1}=\sqrt{2}$$

(b) First evaluating $\mathbf{A}^T\mathbf{A}$ gives

$$\mathbf{A}^T\mathbf{A}=\begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}10&*\\*&20\end{pmatrix}$$

Since the trace is the addition of the leading diagonal elements we don't need to find the entries which are not on the leading diagonal.

$$\sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}=\sqrt{10+20}=\sqrt{30}$$

(c) Note that **B** is a symmetric matrix:

$$\mathbf{B}^T\mathbf{B}=\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\end{pmatrix}\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\end{pmatrix}=\begin{pmatrix}14&&\\&56&\\&&126\end{pmatrix}$$

$$\sqrt{trace\left(\mathbf{B}^T\mathbf{B}\right)}=\sqrt{14+56+126}=\sqrt{196}=14$$

(d) Note that **C** is not a square matrix but we can still find the norm of this matrix.

$$\mathbf{C}^T\mathbf{C}=\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\\4&8&12\\5&10&15\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\2&4&6&8&10\\3&6&9&12&15\end{pmatrix}=\begin{pmatrix}14&&&&\\&56&&&\\&&126&&\\&&&224&\\&&&&350\end{pmatrix}$$

$$\sqrt{trace\left(\mathbf{C}^T\mathbf{C}\right)}=\sqrt{14+56+126+224+350}=\sqrt{770}$$

(iii) Note we have identical answers to parts (i) and (ii). So for all the above examples we have $\|\mathbf{A}\|_F=\sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}$.

(iv) We need to prove this result $\|\mathbf{A}\|_F=\sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}$.

Proof.

Without loss of generality assume $n\ge m$ and $\mathbf{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&\cdots&a_{2n}\\\vdots&\vdots&\vdots&\vdots&\vdots\\a_{m1}&a_{m2}&\cdots&\cdots&a_{mn}\end{pmatrix}$ then

$$\mathbf{A}^T\mathbf{A}=\begin{pmatrix}a_{11}&a_{21}&\cdots&a_{m1}\\a_{12}&a_{22}&\cdots&a_{m2}\\\vdots&\vdots&\vdots&\vdots\\\vdots&\vdots&\vdots&\vdots\\a_{1n}&a_{2n}&&a_{mn}\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&\cdots&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&\cdots&a_{2n}\\\vdots&\vdots&\vdots&\vdots&\vdots\\a_{m1}&a_{m2}&\cdots&\cdots&a_{mn}\end{pmatrix}$$

$$=\begin{pmatrix}a_{11}{}^2+a_{21}{}^2+\cdots+a_{m1}{}^2&*&*&*\\*&a_{12}{}^2+\cdots+a_{m2}{}^2&\cdots&\vdots\\\vdots&*&\ddots&\vdots\\\vdots&\vdots&\cdots&a_{1n}{}^2+\cdots+a_{mn}{}^2\end{pmatrix}$$

Trace of this last matrix is the sum of all the entries in the leading diagonal:

$$trace\left(\mathbf{A}^T\mathbf{A}\right)=\begin{pmatrix} a_{11}^2+a_{21}^2+\cdots+a_{m1}^2 & * & * & * \\ * & a_{12}^2+\cdots+a_{m2}^2 & \cdots & \vdots \\ \vdots & * & \ddots & \vdots \\ \vdots & \vdots & \cdots & a_{1n}^2+\cdots+a_{mn}^2 \end{pmatrix}$$

$$=\left(a_{11}^2+a_{21}^2+\cdots+a_{m1}^2\right)+\left(a_{12}^2+\cdots+a_{m2}^2\right)+\cdots+\left(a_{1n}^2+\cdots+a_{mn}^2\right)$$

$$=a_{11}^2+a_{12}^2+\cdots+a_{mn}^2$$

Taking the square root of both sides gives our required result:

$$\sqrt{trace\left(\mathbf{A}^T\mathbf{A}\right)}=a_{11}^2+a_{12}^2+\cdots+a_{mn}^2=\|\mathbf{A}\|_F$$

39. (i) *How do we show that matrices in S form a subspace of* M_{22}*?*

Use the following result of chapter 3:

Proposition (3.7). A non - empty subset *S* containing vectors **u** and **v** is a subspace of a vector space *V* $\Leftrightarrow$ any linear combination $k\mathbf{u}+c\mathbf{v}$ is also in *S* (k and c are scalars).

Let $\mathbf{A}=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. Also let k_1 and k_2 be scalars. Then consider the linear combination:

$$k_1\mathbf{A}+k_2\mathbf{B}=k_1\begin{pmatrix} a & -b \\ b & a \end{pmatrix}+k_2\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} k_1a & -k_1b \\ k_1b & k_1a \end{pmatrix}+\begin{pmatrix} k_2c & -k_2d \\ k_2d & k_2c \end{pmatrix}$$

$$=\begin{pmatrix} k_1a+k_2c & -\left(k_1b+k_2d\right) \\ k_1b+k_2d & k_1a+k_2c \end{pmatrix}$$

Hence $k_1\mathbf{A}+k_2\mathbf{B}$ is in *S* because matrices in *S* have the form $\mathbf{A}=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Therefore by the above Proposition (3.7) the subset *S* is subspace of M_{22}.

(ii) *What does the term commutative mean?*

$\mathbf{AB}=\mathbf{BA}$. Remember in general matrices do not have this property. Applying matrix multiplication to the above matrices gives:

$$\mathbf{AB}=\begin{pmatrix} ac-bd & -ad-bc \\ bc+ad & ac-bd \end{pmatrix}=\mathbf{BA}$$

40. We use the above proposition repeated here:

Proposition (3.7). A non - empty subset *S* containing vectors **u** and **v** is a subspace of a vector space *V* $\Leftrightarrow$ any linear combination $k\mathbf{u}+c\mathbf{v}$ is also in *S* (k and c are scalars).

Need to show that anti-symmetric matrices are subspace of M_{22}.

Let **A** and **B** be anti-symmetric matrices. *What do we need to prove?*

Required to prove that $k\mathbf{A}+c\mathbf{B}$ where *k* and *c* are scalars is also an anti-symmetric matrix. Consider

$$(k\mathbf{A}+c\mathbf{B})^T = k\mathbf{A}^T + c\mathbf{B}^T = k(-\mathbf{A})+c(-\mathbf{B}) = -(k\mathbf{A}+c\mathbf{B})$$

We have $(k\mathbf{A}+c\mathbf{B})^T = -(k\mathbf{A}+c\mathbf{B})$ therefore $k\mathbf{A}+c\mathbf{B}$ is an anti-symmetric matrix. Hence the set of *n* by *n* anti-symmetric matrices are a subspace of M_{nn}.

41. The solution of the differential equation is $f(x)=c_1\cos(x)+c_2\sin(x)$ where c_1 and c_2 are scalars. This means that $\{\cos(x),\ \sin(x)\}$ span the solution space of $f''(x)+f(x)=0$. We need to show that these are linearly independent to be a basis for this space. Since cos and sin are not multiples of each other so they are linearly independent because

> Proposition (3.12). Two vectors **u** and **v** in a vector space *V* are linearly **dependent** $\Leftrightarrow$ one of the vectors is a multiple of the other.

Hence $\{\cos(x),\ \sin(x)\}$ is a basis for the solution space.