Solutions to Problems on Chapter 6 (Only available to tutors.)

1. Let $\mathbf{A}=\mathbf{I}$ where **I** is the identity matrix then by:

> Proposition (6.16). Let **A** be any *n* by *n* matrix and *k* be a scalar then
>
> $$\det(k\mathbf{A})=k^n\det(\mathbf{A})$$

For $n\geq 2$ we have

$$\det(2\mathbf{A})=2^n\det(\mathbf{A})=2^n\underbrace{\det(\mathbf{I})}_{=1}=2^n$$

However $\det(\mathbf{A})+\det(\mathbf{A})=\det(\mathbf{I})+\det(\mathbf{I})=1+1=2$.

Hence $\det(2\mathbf{A})\neq\det(\mathbf{A})+\det(\mathbf{A})$.

2. We have

$$\det\begin{pmatrix}k&1&1\\1&k&1\\1&1&k\end{pmatrix}=k\det\begin{pmatrix}k&1\\1&k\end{pmatrix}-\det\begin{pmatrix}1&1\\1&k\end{pmatrix}+\det\begin{pmatrix}1&k\\1&1\end{pmatrix}=k^3-3k+2=0$$

Solving the cubic equation $k^3-3k+2=0$:

$$k^3-3k+2=0\ \Rightarrow\ (k+2)(k-1)^2\quad\Rightarrow\quad k=-2,\ 1,\ 1$$

3. (a) Using the following result of question 19 of Exercise 6(c):

$$\det(\mathbf{A}^n)=[\det(\mathbf{A})]^n$$

We have $\det(\mathbf{A}^3)=-2^3=-8$.

(b) Applying the following result of chapter 6:

> Proposition (6.16). Let **A** be any *n* by *n* matrix and *k* be a scalar then $\det(k\mathbf{A})=k^n\det(\mathbf{A})$

We have $\det(5\mathbf{B})=5^6\det(\mathbf{B})=5^6\times 3=46\,875$.

(c) Using the above result of part (a) we have

$$\det\left((\mathbf{AB})^5\right)=[\det(\mathbf{AB})]^5=[\det(\mathbf{A})\det(\mathbf{B})]^5=[-2\times 3]^5=-7776$$

(d) Since we are given $\det(\mathbf{A})=-2$, $\det(\mathbf{B})=3$ so both matrices are invertible because the determinant is *not* zero. Applying the following result of chapter 6:

> Proposition (6.30). If **A** is an invertible (non-singular) matrix then $\det(\mathbf{A}^{-1})=\dfrac{1}{\det(\mathbf{A})}$.

We have

$$\det\left(\mathbf{A}^{-1}\mathbf{B}^{-1}\right)=\det\left(\left(\mathbf{BA}\right)^{-1}\right)$$
$$=\frac{1}{\det\left(\mathbf{BA}\right)}=\frac{1}{\det\left(\mathbf{B}\right)\det\left(\mathbf{A}\right)}=\frac{1}{3\times\left(-2\right)}=-\frac{1}{6}$$

4. We can carry out row operations on the given matrix:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\begin{pmatrix}1001 & 2001 & 3001\\1002 & 2002 & 3002\\1003 & 2003 & 3003\end{pmatrix}$$

Executing

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2-\mathrm{R}_1\\\mathrm{R}_3-\mathrm{R}_2\end{matrix}\begin{pmatrix}1001 & 2001 & 3001\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$$

The determinant of this last matrix and the given matrix is identical. *Why?*
Because we have added (subtracted) a multiple of one row to another and the following result of chapter 6 says the determinants are equal:

(6.23) $$\det\left(\mathbf{B}\right)=\begin{cases}\text{(i)} \quad \det\left(\mathbf{A}\right) & \text{if a multiple of one row is added to another}\\\text{(ii)} \quad -\det\left(\mathbf{A}\right) & \text{if two rows have been interchanged}\\\text{(iii)} \quad k\det\left(\mathbf{A}\right) & \text{if a row has been multiplied by non-zero } k\end{cases}$$

What is the determinant of the last matrix?
Zero because

Proposition (6.10). If a square matrix **A** consists of *two identical rows* then $\det\left(\mathbf{A}\right)=0$.

Hence $\det\left(\mathbf{A}\right)=0$.

5. Using row operations on the given matrix:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\\\mathrm{R}_4\end{matrix}\begin{pmatrix}1 & 2 & 0 & 0\\3 & 4 & 0 & 0\\6 & 7 & 1 & 8\\2 & 1 & 3 & 4\end{pmatrix}=\mathbf{A}$$

Executing the following row operations

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{\dagger}=\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3\\\mathrm{R}_4^{\dagger}=2\mathrm{R}_4-\mathrm{R}_3\end{matrix}\begin{pmatrix}1 & 2 & 0 & 0\\1 & 0 & 0 & 0\\6 & 7 & 1 & 8\\-2 & -5 & 5 & 0\end{pmatrix}$$

Swapping rows over

$$\begin{matrix} R_2^{\dagger} \\ R_1 \\ R_4^{\dagger} \\ R_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & -5 & 5 & 0 \\ 6 & 7 & 1 & 8 \end{pmatrix} = \mathbf{B}$$

The determinant of this last matrix **B** is $\det(\mathbf{B}) = 1\times 2\times 5\times 8 = 80$. *What is the determinant of the given matrix **A**?*

For each row interchange we multiply by -1 but we have two row interchanges which means we multiply by $-1^2 = 1$. The only operation which changes the determinant was doubling the bottom row, $2\mathrm{R}_4 - \mathrm{R}_3$. Hence

$$\det(\mathbf{A}) = \frac{1}{2}\det(\mathbf{B}) = \frac{1}{2}\times 80 = 40$$

6. We have a diagonal matrix so

$$\det\begin{pmatrix} x_1\mathbf{e}_1 & x_2\mathbf{e}_2 & \cdots & x_n\mathbf{e}_n \end{pmatrix} = \det\begin{pmatrix} x_1 & & & \mathbf{0} \\ & x_2 & & \\ & & \ddots & \\ \mathbf{0} & & & x_n \end{pmatrix} = x_1 x_2 \cdots x_n$$

7. Let **A** and **B** be *n* by *n* matrices then

$$\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$$

By the following:

(5.2) $T : U \to V$ is a linear transform $\Leftrightarrow$
(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (b) $T(k\mathbf{u}) = kT(\mathbf{u})$

For the given transform $T(\mathbf{A}) = \det(\mathbf{A})$ part (a) fails so it is not a linear transform.

8. We need to prove the following:

$$\det\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix} \neq 0 \quad \Leftrightarrow \quad \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\} \text{ form a basis for } \mathbb{R}^n$$

Proof.Let $\mathbf{A} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix}$. Then by

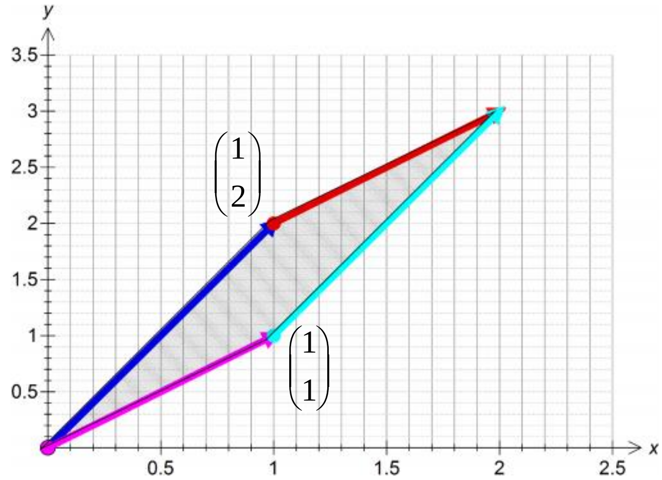
Theorem (6.26). A square matrix **A** is invertible (has an inverse) $\Leftrightarrow$ $\det(\mathbf{A}) \neq 0$

Matrix **A** is invertible. Using the following proposition of chapter 2:

Theorem (2.24). Let $\mathbf{A}$ be a *n* by *n* matrix, then the following statements are equivalent: (a) The matrix **A** is invertible (non-singular). (f) Columns of matrix **A** are linearly independent.

Hence $\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$ are linearly independent so forms a basis for $\mathbb{R}^n$.

9. The shaded area is given by the following determinant:



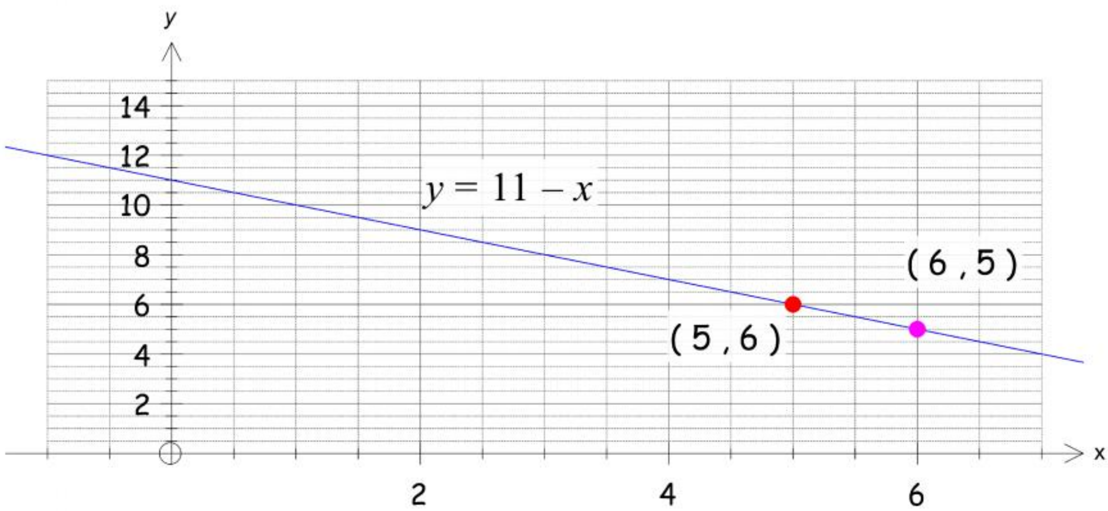
$$\text{Area} = \det\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} = 2-1=1$$

10. We use the given formula for $(5, \ 6)$ and $(6, \ 5)$:

$$\det\begin{pmatrix}x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{pmatrix} = \det\begin{pmatrix}x & y & 1\\ 5 & 6 & 1\\ 6 & 5 & 1\end{pmatrix} = x\det\begin{pmatrix}6 & 1\\ 5 & 1\end{pmatrix} - y\det\begin{pmatrix}5 & 1\\ 6 & 1\end{pmatrix} + \det\begin{pmatrix}5 & 6\\ 6 & 5\end{pmatrix}$$

$$= x(1) - y(-1) - 11 = x + y - 11 = 0$$

The equation is $y = 11 - x$ which is illustrated below:



11. Labelling the rows of the matrix we have

$$\begin{matrix}R_1\\R_2\\R_3\\R_4\end{matrix}\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{pmatrix}$$

Swapping rows $R_4 \rightleftarrows R_1$:

$$\begin{matrix} R_1{*}=R_4 \\ R_2 \\ R_3 \\ R_4{*}=R_1 \end{matrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

then $R_3 \rightleftarrows R_4{*}$:

$$\begin{matrix} R_1{*} \\ R_2 \\ R_3{*}=R_4{*} \\ R_4{**}=R_3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is followed by $R_2 \rightleftarrows R_3{*}$:

$$\begin{matrix} R_1{*} \\ R_2=R_3{*} \\ R_3{**}=R_2 \\ R_4{**}=R_3 \end{matrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}=\mathbf{I}$$

The determinant of the identity matrix is 1. Since we have 3 interchanges so

$$\det(\mathbf{A})=(-1)^3=-1$$

12. If we expand along the first column

$$\det(\mathbf{A})=\det\begin{pmatrix} a & b & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & a & b \\ b & 0 & 0 & 0 & a \end{pmatrix}=a\det\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}+b\det\begin{pmatrix} b & 0 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix}$$

Note that both the matrices on the right are triangular matrices. *How do we find the determinant of triangular matrices?*

Proposition (6.19). The determinant of a triangular or diagonal matrix is the *product* of the entries along the leading diagonal.

The determinant of the 4 by 4 matrices on the Right Hand Side is the product of the entries on the leading diagonal:

$$\det(\mathbf{A})=a\det\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}+b\det\begin{pmatrix} b & 0 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix}=a\left[a^4\right]+b\left[b^4\right]=a^5+b^5$$

13. (i) We have

$$\det\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 0 - 1 = -1$$

$$\det\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} = (-1)^1 = -1 \quad [\text{Interchanging top and bottom rows}]$$

$$\det\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} = (-1)^2 = 1 \quad \begin{bmatrix}\text{Switching top and bottom rows}\\ \text{and the middle rows}\end{bmatrix}$$

Similarly we have $\det(\mathbf{A}_5) = 1,\ \det(\mathbf{A}_6) = -1$

(ii) We have

$$\det(\mathbf{A}_2) = \det(\mathbf{A}_3) = -1,\ \det(\mathbf{A}_4) = \det(\mathbf{A}_5) = 1,\ \det(\mathbf{A}_6) = -1$$

We need to define what is meant by the floor function.

The floor function denoted $\lfloor x \rfloor$ is defined as the largest integer less than or equal to *x*:

$$\lfloor x \rfloor = a \quad \text{where } a \in \mathbb{Z} \text{ and } a \le x$$

For example $\lfloor 4 \rfloor = 4,\ \lfloor 4.5 \rfloor = 4,\ \lfloor ® \rfloor = 3,\ \lfloor e \rfloor = 2$.

The formula for $\det(\mathbf{A}_n) = (-1)^{\lfloor n/2 \rfloor}$.

14. The determinant of the given matrix is:

$$\det(\mathbf{A}) = \det\begin{pmatrix}13 & 4\\ 1 & 1\end{pmatrix} = 13 - 4 = 9$$

The trace of matrix **A** is sum of the leading entries:

$$trace(\mathbf{A}) = 13 + 1 = 14$$

Substituting these into $\mathbf{X} = \dfrac{1}{\sqrt{trace(\mathbf{A}) + 2\sqrt{\det(\mathbf{A})}}}\left[\mathbf{A} + \left(\sqrt{\det(\mathbf{A})} \times \mathbf{I}\right)\right]$ gives

$$\mathbf{X} = \frac{1}{\sqrt{14 + \left(2 \times \sqrt{9}\right)}}\left[\begin{pmatrix}13 & 4\\ 1 & 1\end{pmatrix} + \left(\sqrt{9} \times \mathbf{I}\right)\right]$$

$$= \frac{1}{\sqrt{14+6}}\left[\begin{pmatrix}13 & 4\\ 1 & 1\end{pmatrix} + \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\right] = \frac{1}{\sqrt{20}}\left[\begin{pmatrix}16 & 4\\ 1 & 4\end{pmatrix}\right]$$

Squaring the last matrix gives

$$\mathbf{X}^2 = \frac{1}{\sqrt{20}}\left[\begin{pmatrix}16 & 4\\ 1 & 4\end{pmatrix}\right]\frac{1}{\sqrt{20}}\left[\begin{pmatrix}16 & 4\\ 1 & 4\end{pmatrix}\right]$$

$$= \frac{1}{20}\begin{pmatrix}260 & 80\\ 20 & 20\end{pmatrix} = \begin{pmatrix}13 & 4\\ 1 & 1\end{pmatrix} = \mathbf{A}$$

Note that $\mathbf{X}^2 = \mathbf{A}$ which means the formula **X** gives the square root of matrix **A**.

15. (i) You have to carry out the matrix multiplication of row by column or use Matlab, Maple.

(ii) The determinants of lower **L** and upper triangular matrices is 1 because by

> Proposition (6.19). The determinant of a triangular or diagonal matrix is the *product* of the entries along the leading diagonal.

We have $\det(\mathbf{L}) = \det(\mathbf{U}) = 1\times1\times1\times1\times1 = 1$.

How do we find the determinant of the Pascal matrix $\mathbf{P}_5$ *?*

Well we are given that $\mathbf{P}_5 = \mathbf{LU}$ and by:

> Proposition (6.28). If **A** and **B** are square matrices of the same size then
>
> $$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Applying this result to $\mathbf{P}_5 = \mathbf{LU}$ gives

$$\det(\mathbf{P}_5) = \det(\mathbf{LU}) = \det(\mathbf{L})\det(\mathbf{U}) = 1\times1 = 1$$

16. Consider $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$. Since matrix **A** is a triangular matrix so its determinant is the product of the entries along the leading diagonal: $\det(\mathbf{A}) = 3\times4\times0 = 0$.

17. The result is false because any triangular matrix with zeros on the leading diagonal will have a determinant of zero because:

> Proposition (6.19). The determinant of a triangular or diagonal matrix is the *product* of the entries along the leading diagonal.

For example $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 9 \\ 0 & 0 & 7 \end{pmatrix}$ has a determinant of zero which means it is singular (non-invertible).

18. (i) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & 3 \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} 2 & 11 \\ 3 & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{pmatrix} 0 & 15 \\ 1 & 11 \end{pmatrix}$$

We have

$$\mathbf{AB}-\mathbf{BA}=\begin{pmatrix}2 & -4\\ 2 & -2\end{pmatrix} \text{ and } \det\left[\mathbf{AB}-\mathbf{BA}\right]=\det\begin{pmatrix}2 & -4\\ 2 & -2\end{pmatrix}=-4+8=4\neq 0$$

(ii) We need to prove $\det(\mathbf{AB})-\det(\mathbf{BA})=0$.

Proof.

By the following proposition of chapter 6:

(6.28)	$\det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})$

We have

$$\det(\mathbf{AB})-\det(\mathbf{BA})=\det(\mathbf{A})\det(\mathbf{B})-\det(\mathbf{B})\det(\mathbf{A})=0$$

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19. No because both $\mathbf{A}=\begin{pmatrix}1 & 5\\ 0 & 0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}0 & 0\\ 1 & 3\end{pmatrix}$ have a determinant of zero but

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1 & 5\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 1 & 3\end{pmatrix}=\begin{pmatrix}1 & 5\\ 1 & 3\end{pmatrix},\quad \det(\mathbf{A}+\mathbf{B})=\det\begin{pmatrix}1 & 5\\ 1 & 3\end{pmatrix}=3-5=-2$$

Since $\mathbf{A}+\mathbf{B}$ is not a member of the set. The set is *not* closed under matrix addition so cannot be a vector space.

20. *Proof*.

Let **T** be a triangular matrix with non-zero entries on the leading diagonal.

By the following Proposition of chapter 6:

> Proposition (6.19). The determinant of a triangular or diagonal matrix is the *product* of the entries along the leading diagonal.

We have $\det(\mathbf{T})\neq 0$ because none of the leading diagonal entries are zero. By

> Theorem (6.26). A square matrix **A** is invertible (has an inverse) $\Leftrightarrow$ $\det(\mathbf{A})\neq 0$

Hence matrix **T** is invertible.

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21. Consider the triangular matrix $\mathbf{T}=\begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 5\\ 0 & 0 & 7\end{pmatrix}$ then $\det(\mathbf{T})=1\times 0\times 7=0$. By the above

Theorem (6.26) we conclude that **T** is non-invertible. Hence a triangular matrix may not be invertible.

22. The Wronskian is given by $W(f,\quad g)=\det\begin{pmatrix}f(x) & g(x)\\ f'(x) & g'(x)\end{pmatrix}$. In our case we have

$$W\left(e^{2x}\cos(x),\quad e^{2x}\sin(x)\right)=\det\begin{pmatrix} e^{2x}\cos(x) & e^{2x}\sin(x) \\ \left[e^{2x}\cos(x)\right]' & \left[e^{2x}\sin(x)\right]' \end{pmatrix} \qquad (*)$$

Using the product rule for differentiating the functions on the bottom row we have

$$\left[e^{2x}\cos(x)\right]'=2e^{2x}\cos(x)-e^{2x}\sin(x)=e^{2x}\left(2\cos(x)-\sin(x)\right)$$

$$\left[e^{2x}\sin(x)\right]'=2e^{2x}\sin(x)+e^{2x}\cos(x)=e^{2x}\left(2\sin(x)+\cos(x)\right)$$

Putting these into (*) gives

$$\begin{aligned} W\left(e^{2x}\cos(x),\quad e^{2x}\sin(x)\right)&=\det\begin{pmatrix} e^{2x}\cos(x) & e^{2x}\sin(x) \\ e^{2x}\left(2\cos(x)-\sin(x)\right) & e^{2x}\left(2\sin(x)+\cos(x)\right) \end{pmatrix} \\ &=e^{2x}\cos(x)\left[e^{2x}\left(2\sin(x)+\cos(x)\right)\right] \\ &\qquad -e^{2x}\sin(x)\left[e^{2x}\left(2\cos(x)-\sin(x)\right)\right] \\ &=e^{4x}\left[2\sin(x)\cos(x)+\cos^2(x)-2\sin(x)\cos(x)+\sin^2(x)\right] \\ &=e^{4x}\underbrace{\left[\cos^2(x)+\sin^2(x)\right]}_{=1}=e^{4x} \end{aligned}$$

Hence $W\left(e^{2x}\cos(x),\quad e^{2x}\sin(x)\right)=e^{4x}\neq 0$.

23. The error is the second line where

$$\left[\det(\mathbf{A}+\mathbf{B})\right]^2=\left[\det(\mathbf{A})+\det(\mathbf{B})\right]^2$$

Because $\det(\mathbf{A}+\mathbf{B})\neq\det(\mathbf{A})+\det(\mathbf{B})$.

24. Applying row operations to the given matrix yields

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}$$

Executing the row operations:

$$\begin{matrix} R_1 \\ R_2{*}=R_2-R_1 \\ R_3 \\ R_4{*}=R_4-R_3 \end{matrix}\begin{pmatrix} x & 1 & 1 & 1 \\ 1-x & x-1 & 0 & 0 \\ 1 & 1 & x & 1 \\ 0 & 0 & 1-x & x-1 \end{pmatrix}$$

Carrying out the following row operations:

$$\begin{matrix} R_1 \\ R_2{*} \\ R_3-R_1 \\ R_4{*} \end{matrix}\begin{pmatrix} x & 1 & 1 & 1 \\ 1-x & x-1 & 0 & 0 \\ 1-x & 0 & x-1 & 0 \\ 0 & 0 & 1-x & x-1 \end{pmatrix}$$

We find the determinant by using the bottom row:

$$\det\begin{pmatrix} x & 1 & 1 & 1 \\ 1-x & x-1 & 0 & 0 \\ 1-x & 0 & x-1 & 0 \\ 0 & 0 & 1-x & x-1 \end{pmatrix} = -(1-x)\det\begin{pmatrix} x & 1 & 1 \\ 1-x & x-1 & 0 \\ 1-x & 0 & 0 \end{pmatrix} + (x-1)\det\begin{pmatrix} x & 1 & 1 \\ 1-x & x-1 & 0 \\ 1-x & 0 & x-1 \end{pmatrix}$$

$$= -(1-x)\det\begin{pmatrix} 1-x & x-1 \\ 1-x & 0 \end{pmatrix} + (x-1)\left[\det\begin{pmatrix} 1-x & x-1 \\ 1-x & 0 \end{pmatrix} + (x-1)\det\begin{pmatrix} x & 1 \\ 1-x & x-1 \end{pmatrix}\right]$$

$$= -(1-x)\left[0-(1-x)(x-1)\right] + (x-1)\left[-(1-x)(x-1) + (x-1)\left[x(x-1)-(1-x)\right]\right]$$

$$= (x-1)^3 + (x-1)\left[(x-1)^2 + (x-1)^2\left[x+1\right]\right]$$

$$= (x-1)^3 + (x-1)^3\left[1+\left[x+1\right]\right] = (x-1)^3\left[1+1+x+1\right] = (x-1)^3\left[3+x\right]$$

None of the row operations affect the determinant because by chapter 6 we have:

(6.23) $$\det(\mathbf{B}) = \begin{cases} \text{(i)} \quad \det(\mathbf{A}) & \text{if a multiple of one row is added to another} \\ \text{(ii)} \quad -\det(\mathbf{A}) & \text{if two rows have been interchanged} \\ \text{(iii)} \quad k\det(\mathbf{A}) & \text{if a row has been multiplied by non-zero } k \end{cases}$$

We have only added a multiple of one row onto another.

Hence $\det(\mathbf{A}) = (x-1)^3(3+x)$.

(ii) The matrix is non-invertible if $\det(\mathbf{A}) = 0$. Hence we need to find *x* values such that

$$\det(\mathbf{A}) = (x-1)^3(3+x) = 0 \quad \Rightarrow \quad x=1, \ x=-3$$

25. (i) By the text of chapter 6 we have the following result:

Proposition (6.14). Let **A** be a square matrix then $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$.

Proof.

The matrix under consideration is an anti-symmetric matrix therefore $\mathbf{A}^T = -\mathbf{A}$. We have

$$\det\left(\mathbf{A}^T\right) = \det(-\mathbf{A}) = (-1)^n\det(\mathbf{A}) \qquad \left[\text{By (6.16)} \quad \det(k\mathbf{A}) = k^n\det(\mathbf{A})\right]$$

$$= \det(\mathbf{A}) \qquad \left[\text{By (6.14)}\right]$$

The last two lines gives $(-1)^n\det(\mathbf{A}) = \det(\mathbf{A})$. If *n* is odd then

$$-\det(\mathbf{A}) = \det(\mathbf{A}) \quad \Rightarrow \quad -2\det(\mathbf{A}) = 0 \quad \Rightarrow \quad \det(\mathbf{A}) = 0$$

If *n* is odd then the determinant of anti-symmetric matrix is zero.

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(ii) If matrix **A** is *n* by *n* where *n* is odd then by result (i) $\det(\mathbf{A}) = 0$.

$$\det\left(\mathbf{A}^m\mathbf{A}^T\right) = \det\left(\mathbf{A}^m\right)\det\left(\mathbf{A}^T\right)$$

$$= \left[\det(\mathbf{A})\right]^m\det(\mathbf{A}) \qquad \left[\text{Because} \quad \det\left(\mathbf{A}^T\right) = \det(\mathbf{A})\right]$$

$$= \left[\det(\mathbf{A})\right]^{m+1}$$

$$= \begin{cases} \left[\det(\mathbf{A})\right]^{m+1} & \text{if } m+1 \text{ is even} \\ 0 & \text{if } m+1 \text{ is odd} \end{cases}$$