Solutions to Problems on Chapter 7 (Only available to tutors.)

1. (a) Since the given matrix $\mathbf{A}=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$ is triangular so the eigenvalues are the leading diagonal entries; $a_1=1$ and $a_2=-1$. The characteristic equation is given by

$$(a+1)(a-1)=a^2-1=0$$

By the Cayley Hamilton Theorem:

> Cayley Hamilton Theorem (7.11). Every square matrix **A** is a root of the characteristic equation, that is $p(\mathbf{A})=\mathbf{O}$.

We have $p(a)=a^2-1=0$ so

$$p(\mathbf{A})=\mathbf{A}^2-\mathbf{I}=\mathbf{O}\quad\Rightarrow\quad\mathbf{A}^2=\mathbf{I}$$

Hence $\mathbf{A}^2=\mathbf{AA}=\mathbf{I}\;\Rightarrow\;\mathbf{A}^{-1}=\mathbf{A}$.

(b) The characteristic equation of $\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}$ is given by $a^2-3=0$:

$$\mathbf{A}^2=3\mathbf{I}$$

Rearranging this we have

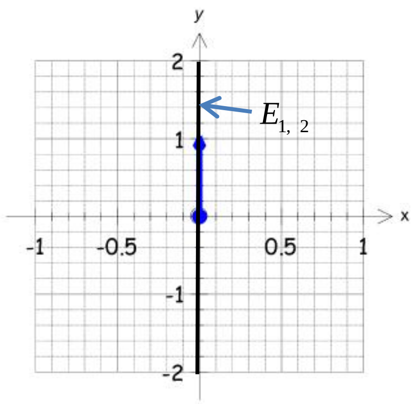
$$\mathbf{A}\left(\frac{1}{3}\mathbf{A}\right)=\mathbf{I}\quad\Rightarrow\quad\mathbf{A}^{-1}=\frac{1}{3}\mathbf{A}=\frac{1}{3}\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}$$

2. Since the given matrix $\mathbf{A}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ is a triangular matrix so the eigenvalues of **A** are $a_{1,\,2}=0$. Let **u** be the given vector belonging to $a_{1,\,2}=0$ then

$$(\mathbf{A}-0\mathbf{I})\mathbf{u}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\Rightarrow\quad\begin{matrix}0+y=0\\ 0+0=0\end{matrix}$$

From the first equation we must have $y=0$. Hence *x* can be any non-zero real number. So $\mathbf{u}=s\begin{pmatrix}1\\ 0\end{pmatrix}$ where $s\neq 0$ is the general eigenvector belonging to $a_{1,\,2}=0$.

The eigenspace $E_{1,\,2}$ is the



A basis for the eigenspace is $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

3. Since the given matrix **A** is a triangular matrix so the eigenvalues are the entries on the leading diagonal:

> Proposition (7.6). If an *n* by *n* matrix **A** is a diagonal or triangular matrix then the eigenvalues of **A** are the entries along the leading diagonal.

The eigenvalues of matrix **A** are $a_{1,2,3} = 1$. Let **u** be the eigenvector belonging to $a_{1,2,3} = 1$:

$$(\mathbf{A}-\mathbf{I})\mathbf{u} = \begin{pmatrix}1-1 & 2 & 3\\ 0 & 1-1 & 4\\ 0 & 0 & 1-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0 & 2 & 3\\ 0 & 0 & 4\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

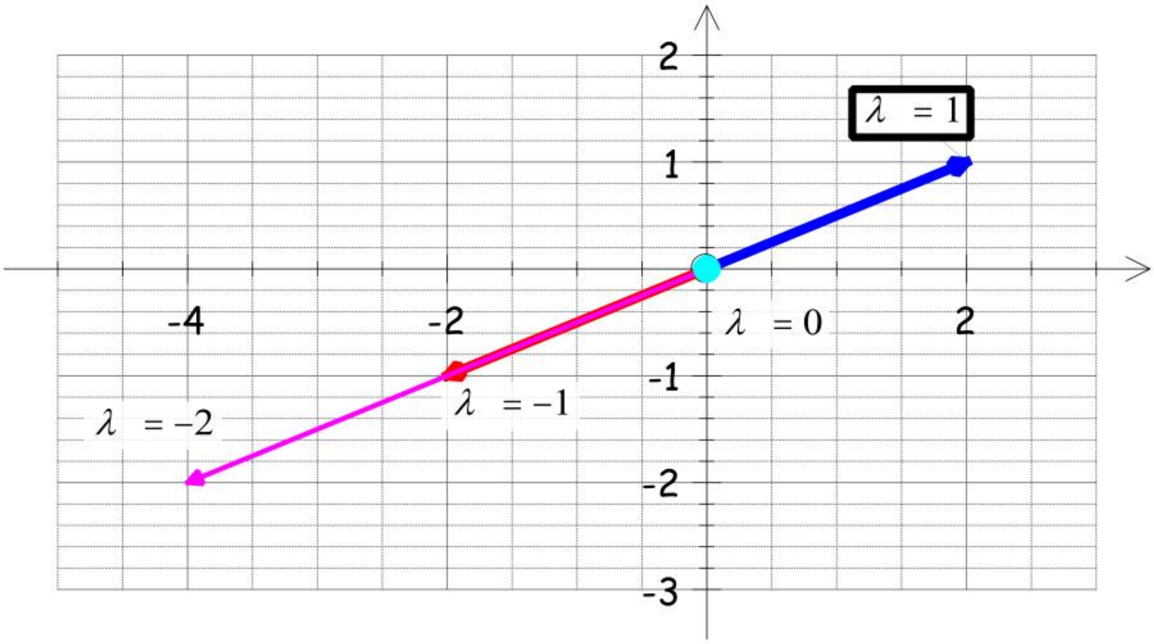
From the top two rows we have

$$y = z = 0$$

Also x can be any non-zero real number, $x = s \neq 0$ where *s* is any real number. The general eigenvector is $\mathbf{u} = s\begin{pmatrix}1\\0\\0\end{pmatrix}$ where $s \in \mathbb{R}$ and $s \neq 0$.

4. (a) The matrix **A** stretches the length of **x** by a factor of 5 in the same direction.
(b) The matrix **A** stretches the length of **x** by a factor of 2 in the opposite direction.

5. For (i), (ii), (iii) and (iv):



6. Remember the trace of the matrix gives the sum of the eigenvalues:

(7.9)	$tr(\mathbf{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

The trace of matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $1+4=5$. Therefore

$$5.74 + \lambda = 5 \quad \Rightarrow \quad \lambda = -0.74$$

The other eigenvalue is -0.74.

7. The matrix $\mathbf{A} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is an upper triangular matrix with eigenvalues $\lambda_1 = a$ and $\lambda_2 = c$. Evaluating $\mathbf{Ax}$ gives

$$\mathbf{Ax} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence $\lambda_1 = a$ has the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

8. Evaluating $\mathbf{Ax}$ gives

$$\mathbf{Ax} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 3/2 \\ -1 \end{pmatrix}$$

Hence **x** is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

9. *Proof*.

The definition of eigenvalues and eigenvectors from chapter 7 is:

(7.1) $\mathbf{Au} = \lambda\mathbf{u}$

Since **u** is an eigenvector of **A** with eigenvalue λ so we have

$$\mathbf{Au} = \lambda\mathbf{u}$$

Multiplying this by the non-zero scalar *c* gives

$$\mathbf{A}(c\mathbf{u}) = \lambda(c\mathbf{u})$$

Hence by the above definition (7.1) $c\mathbf{u}$ is an eigenvector belonging to λ.

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10. Because if **u** was an eigenvector of matrix **A** then it would lay in the same or exactly in the opposite direction (180° out of phase).

11. (i) Using the formula for evaluating the eigenvalues λ:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}$$
$$= (2-\lambda)(3-\lambda) - 2$$
$$= \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0 \quad \Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = 4$$

(ii) Remember matrix **P** is the eigenvector matrix which means we need to find the eigenvectors of matrix **A**. Let **u** be the eigenvector belonging to $\lambda_1 = 1$. Then

$$\begin{pmatrix} 2-1 & 1 \\ 2 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the simultaneous equations gives

$$x + y = 0 \quad \Rightarrow \quad x = -y$$

Let $y = 1$ then $x = -1$ so $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Let **v** be the eigenvector belonging to $\lambda_2 = 4$:

$$\begin{pmatrix} 2-4 & 1 \\ 2 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -2x + y = 0 \\ 2x - y = 0 \end{matrix}$$

From the top equation we have

$$y = 2x$$

Let $x = 1$ then $y = 2x = 2(1) = 2$. Our eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Hence our eigenvector matrix $\mathbf{P} = (\mathbf{u} \quad \mathbf{v}) = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$.

(iii) From Chapter 7 section C we have $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}'$ where **D′** is the diagonal eigenvalue matrix; $\mathbf{D}' = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. We need to find the diagonal matrix **D** such that $\mathbf{D}^2 = \mathbf{D}'$. Hence

$$\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \text{ because } \mathbf{D}^2=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$$

(iv) Using the given result:

$$\mathbf{B}=\mathbf{PDP}^{-1} \qquad (*)$$

What is $\mathbf{P}^{-1}$ *equal to?*

$$\mathbf{P}^{-1}=\frac{1}{\det(\mathbf{P})}\begin{pmatrix}2 & -1\\ -1 & -1\end{pmatrix}=-\frac{1}{3}\begin{pmatrix}2 & -1\\ -1 & -1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}$$

Evaluating the matrix **B** by substituting $\mathbf{P}=\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ and

$\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}$ into (*) gives

$$\mathbf{B}=\mathbf{PDP}^{-1}=\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\frac{1}{3}\begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}$$
$$=\frac{1}{3}\begin{pmatrix}-1 & 2\\ 1 & 4\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}4 & 1\\ 2 & 5\end{pmatrix}$$

Evaluating $\mathbf{B}^2=\frac{1}{3}\begin{pmatrix}4 & 1\\ 2 & 5\end{pmatrix}\frac{1}{3}\begin{pmatrix}4 & 1\\ 2 & 5\end{pmatrix}=\frac{1}{9}\begin{pmatrix}18 & 9\\ 18 & 27\end{pmatrix}=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}=\mathbf{A}$.

12. The eigenvalues of $\mathbf{A}=\begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}$ are $\lambda_1=1,\ \lambda_2=9$ and corresponding eigenvectors are $\mathbf{u}=\begin{pmatrix}1\\ 1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\ -1\end{pmatrix}$. Hence the eigenvector matrix **P** and its inverse $\mathbf{P}^{-1}$ are

$$\mathbf{P}=(\mathbf{u}\quad \mathbf{v})=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \text{ and } \mathbf{P}^{-1}=-\frac{1}{2}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\frac{1}{2}\mathbf{P}$$

Using the formula $\mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}$ where $\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 9\end{pmatrix}$ is the eigenvalue matrix; we have

$$\mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1^n & 0\\ 0 & 9^n\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 9^n\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Evaluating with $n=\frac{1}{2}$ gives

$$\mathbf{A}^{1/2} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 9^{1/2}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & -3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4 & -2\\ -2 & 4\end{pmatrix} = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$$

13. Let $\mathbf{B} = \sqrt{\mathbf{A}} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then

$$\mathbf{B}^2 = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$
$$= \begin{pmatrix}a^2 + bc & ab + bd\\ ac + dc & bc + d^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Equating entries

$$a^2 + bc = 1,\ (a+d)b = 0,\ (a+d)c = 0,\ bc + d^2 = 1$$

One solution is $a = d = 0,\ b = c = 1$. Therefore $\mathbf{B} = \sqrt{\mathbf{A}} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

14. The eigenvalues of matrix $\mathbf{A} = \begin{pmatrix}0 & 4\\ -1 & 4\end{pmatrix}$ are $a_{1,\,2} = 2$ which means we have repeated eigenvalues. The corresponding eigenvector is $\begin{pmatrix}2\\ 1\end{pmatrix}$ and there is no other linearly independent eigenvector. This means the eigenvector matrix $\mathbf{P} = \begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}$ which is **not** invertible, so we cannot diagonalize the given matrix **A**.

15. We first find the eigenvalues a :

$$\det(\mathbf{A} - a\mathbf{I}) = \det\begin{pmatrix}4 - a & 3\\ 1 & 2 - a\end{pmatrix}$$
$$= (a - 4)(a - 2) - 3$$
$$= a^2 - 6a + 5 = (a - 5)(a - 1) = 0 \quad \Rightarrow \quad a_1 = 5,\ a_2 = 1$$

Let **u** be the eigenvector belonging to $a_1 = 5$:

$$\begin{pmatrix}4-5 & 3\\ 1 & 2-5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1 & 3\\ 1 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{pmatrix}3\\ 1\end{pmatrix} \quad [\text{Because } x = 3,\ y = 1]$$

Let **v** be the eigenvector belonging to $a_2 = 1$:

$$\begin{pmatrix}4-1 & 3\\ 1 & 2-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}3 & 3\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{pmatrix}1\\ -1\end{pmatrix} \quad [\text{Because } x = 1,\ y = -1]$$

The eigenvector matrix **P** is given by $\mathbf{P}=(\mathbf{u}\quad \mathbf{v})=\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}5 & 0\\ 0 & 1\end{pmatrix}$.

We determine $\mathbf{A}^5$ by using the following of chapter 7:

$$\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

Let $m=5$ then $\mathbf{A}^5=\mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$. Evaluating $\mathbf{P}^{-1}$:

$$\mathbf{P}^{-1}=\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}^{-1}=\frac{1}{-3-1}\begin{pmatrix}-1 & -1\\ -1 & 3\end{pmatrix}=-\frac{1}{4}\begin{pmatrix}-1 & -1\\ -1 & 3\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}$$

Substituting the various components of $\mathbf{A}^5=\mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ gives

$$\begin{aligned}\mathbf{A}^5=\mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}&=\frac{1}{4}\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 1\end{pmatrix}^5\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}5^5 & 0\\ 0 & 1^5\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3125 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3125 & 3125\\ 1 & -3\end{pmatrix}\\ &=\frac{1}{4}\begin{pmatrix}9376 & 9372\\ 3124 & 3128\end{pmatrix}=\begin{pmatrix}2344 & 2343\\ 781 & 782\end{pmatrix}\end{aligned}$$

16. The given matrix $\mathbf{A}=\begin{pmatrix}1 & 5 & 7\\ 0 & 2 & 5\\ 0 & 0 & 3\end{pmatrix}$ is triangular so the eigenvalues are the entries on the leading diagonal which are 1 , 2 and 3. Hence $\lambda_1=1,\ \lambda_2=2$ and $\lambda_3=3$ so the characteristic equation is given by

$$(\lambda-1)(\lambda-2)(\lambda-3)=\lambda^3-6\lambda^2+11\lambda-6$$

By the Cayley Hamilton Theorem (7.11) we have

$$\mathbf{A}^3-6\mathbf{A}^2+11\mathbf{A}-6\mathbf{I}=\mathbf{O}$$

17. We need to use the following two results of chapter 7:

Proposition (7.14). Let **A** and **B** be similar matrices. The eigenvalues of these matrices are identical.

And

Proposition (7.9). Let **A** be any *n* by *n* matrix with eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots\ \lambda_n$:

(a) The determinant of the matrix **A** is given by $\det(\mathbf{A}) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n$.

Proof.
Since we are given that the matrices **A** and **B** are similar so by (7.14) they have the same eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$. By Proposition (7.9) we have

$$\det(\mathbf{A}) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n = \det(\mathbf{B})$$

This completes our proof.

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18. *Proof*.
We are given that the null space of matrix **A** is the zero vector. Applying the following to matrix **A**:

Proposition (3.39). Let **A** be an *n* by *n* matrix then the following are equivalent:

(a) Matrix **A** is invertible
(c) Null space of matrix **A** only contains the zero vector.

Therefore matrix **A** is invertible. By the following proposition:

Proposition (7.7). A square matrix **A** is invertible (has an inverse) $\Leftrightarrow$ $\lambda = 0$ is *not* an eigenvalue of the matrix **A**.

$\lambda = 0$ is *not* an eigenvalue of the matrix **A**. This is our required result.

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19. Since the characteristic polynomial is $(\lambda^2 - 4)(\lambda + 3)$ so the eigenvalues are the roots of the equation

$$(\lambda^2 - 4)(\lambda + 3) = 0 \quad \Rightarrow \quad \lambda^2 - 4 = 0,\ \lambda + 3 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda + 2) = 0,\ \lambda = -3$$

Hence the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = -2$ and $\lambda_3 = -3$. A triangular matrix with these eigenvalues as its entries on the leading diagonal will have the given characteristic polynomial. An example is

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 5 & -2 & 0 \\ 7 & 8 & -3 \end{pmatrix}$$

20. Let $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ be the eigenvalues of matrix **A**. The characteristic polynomial $p(\lambda)$ is given by

$$\begin{aligned} p(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3) &= (\lambda^2 - 3\lambda + 2)(\lambda - 3) \\ &= \lambda^3 - 3\lambda^2 + 2\lambda - 3\lambda^2 + 9\lambda - 6 \\ &= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \end{aligned}$$

By the Cayley Hamilton Theorem we have

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} = 6\mathbf{I}$$
$$\mathbf{A}\left(\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}\right) = 6\mathbf{I}$$
$$\mathbf{A}\underbrace{\left[\frac{1}{6}\left(\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}\right)\right]}_{=\mathbf{A}^{-1}} = \mathbf{I}$$

Hence $\mathbf{A}^{-1} = \frac{1}{6}\left(\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I}\right)$.

21. For the given matrix $\mathbf{A} = \begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$ the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 0.5$ because we have (upper) triangular matrix.

Let **u** be the eigenvector belonging to $\lambda_1 = 1$:

$$(\mathbf{A} - \mathbf{I})\mathbf{u} = \begin{pmatrix} 0 & 0.5 \\ 0 & -0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let **v** be the eigenvector belonging to $\lambda_2 = 0.5$:

$$(\mathbf{A} - 0.5\mathbf{I})\mathbf{v} = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Our invertible matrix $\mathbf{P} = (\mathbf{u} \quad \mathbf{v}) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{P}^{-1} = -\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \mathbf{P}$.

Using the following formula for evaluating the power of a matrix from chapter 7:

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

The diagonal matrix **D** is the eigenvalue matrix; $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$. *What is $\mathbf{D}^m$ equal to?*

$$\mathbf{D}^m = \begin{pmatrix} 1^m & 0 \\ 0 & 0.5^m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5^m \end{pmatrix}$$

Since 0.5 is less than 1 so using the result given in the question, $\lim_{m\to\infty}\left(0.5^m\right) = 0$. Hence we have

$$\lim_{m\to\infty}\left(\mathbf{D}^m\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Substituting this into $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ gives

$$\lim_{m\to\infty}\left(\mathbf{A}^m\right) = \lim_{m\to\infty}\left(\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}\right) = \mathbf{P}\left(\lim_{m\to\infty}\left(\mathbf{D}^m\right)\right)\mathbf{P}^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore $\lim_{n\to\infty}\left(\mathbf{A}^n\right)\mathbf{x}=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}2\\ 0\end{pmatrix}$.

22. We need to find the vector $\mathbf{x}_n=\mathbf{L}^n\mathbf{x}_0$ as $n\to\infty$ which means we need to evaluate the matrix $\mathbf{L}^n$ as $n\to\infty$. Since we are given three distinct eigenvalues so the matrix **A** is diagonalizable and the eigenvalue diagonal matrix **D** is given by

$$\mathbf{D}=\begin{pmatrix}0.91 & 0 & 0\\ 0 & -0.88 & 0\\ 0 & 0 & -0.03\end{pmatrix}$$

Let **P** be the eigenvector matrix so by

$$\mathbf{A}^m=\mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

We have

$$\mathbf{L}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}=\mathbf{P}\begin{pmatrix}0.91 & 0 & 0\\ 0 & -0.88 & 0\\ 0 & 0 & -0.03\end{pmatrix}^n\mathbf{P}^{-1}$$

$$=\mathbf{P}\begin{pmatrix}0.91^n & 0 & 0\\ 0 & -0.88^n & 0\\ 0 & 0 & -0.03^n\end{pmatrix}\mathbf{P}^{-1}$$

As $n\to\infty$ the entries $0.91^n\to 0,\ -0.88^n\to 0$ and $-0.03^n\to 0$ because of the given result in the hint of the question.

Therefore as $n\to\infty$ we have

$$\mathbf{L}^n=\mathbf{P}\begin{pmatrix}0.91^n & 0 & 0\\ 0 & -0.88^n & 0\\ 0 & 0 & -0.03^n\end{pmatrix}\mathbf{P}^{-1}=\mathbf{P}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\mathbf{P}^{-1}=\mathbf{P}\mathbf{O}\mathbf{P}^{-1}=\mathbf{O}$$

Hence as $n\to\infty$, $\mathbf{L}^n\to\mathbf{O}$ so $\mathbf{x}_n=\mathbf{L}^n\mathbf{x}_0=\mathbf{O}_{nn}\mathbf{x}_0=\mathbf{O}_{n1}$. In the long term the population will vanish.

23. *Proof*.

We are given that matrix **A** has eigenvalues a_1 and a_2 with the corresponding eigenvectors **u** and **v**. This means we have

$$\mathbf{A}\mathbf{u}=a_1\mathbf{u}\quad\text{and}\quad\mathbf{A}\mathbf{v}=a_2\mathbf{v}$$

Using the following result of chapter 7:

Proposition (7.8). Let **A** be a square matrix with eigenvector **u** belonging to eigenvalue a.

(a) If *m* is a natural number then λ^m is an eigenvalue of the matrix $\mathbf{A}^m$ with the *same* eigenvector **u**.

We have for the natural number *n*:

$$\mathbf{A}^n\mathbf{u} = \lambda_1^n\mathbf{u} \quad \text{and} \quad \mathbf{A}^n\mathbf{v} = \lambda_2^n\mathbf{v}$$

We are given that the vector $\mathbf{w} = c\mathbf{u} + k\mathbf{v}$ so

$$\begin{aligned}\mathbf{A}^n\mathbf{w} &= \mathbf{A}^n\left(c\mathbf{u} + k\mathbf{v}\right)\\ &= c\mathbf{A}^n\mathbf{u} + k\mathbf{A}^n\mathbf{v} = c\lambda_1^n\mathbf{u} + k\lambda_2^n\mathbf{v} \quad \left[\text{From above}\right]\end{aligned}$$

Hence we have our result $\mathbf{A}^n\mathbf{w} = c\lambda_1^n\mathbf{u} + k\lambda_2^n\mathbf{v}$.

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For the given matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ the eigenvalues and eigenvectors are

$$\lambda_1 = 1,\ \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 2,\ \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

We need to write the given vector $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of vectors **u** and **v**:

$$c\mathbf{u} + k\mathbf{v} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad k = 1, \quad c = -2$$

Substituting $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, $n = 10$, $k = 1$, $c = -2$, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\lambda_1 = 1$, $\lambda_2 = 2$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ into the formula $\mathbf{A}^n\mathbf{w} = c\lambda_1^n\mathbf{u} + k\lambda_2^n\mathbf{v}$ gives

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^{10}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\left(1\right)^{10}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\left(2^{10}\right)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3072 \\ 1024 \end{pmatrix} = \begin{pmatrix} 3070 \\ 1024 \end{pmatrix}$$

24. The eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$ are given by

$$\lambda_1 = 5,\ \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 2,\ \mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

(a) We need to write the vector $\mathbf{u}_0 = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ as a linear combination of the eigenvectors:

$$c\mathbf{u} + k\mathbf{v} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \quad \Rightarrow \quad c = 100,\ k = 0$$

Substituting various components into the formula $\mathbf{A}^n\mathbf{w} = c\lambda_1^n\mathbf{u} + k\lambda_2^n\mathbf{v}$ gives

$$\mathbf{A}^n\begin{pmatrix}100\\100\end{pmatrix}=100\left(5^n\right)\begin{pmatrix}1\\1\end{pmatrix}+0\left(2^n\right)\begin{pmatrix}1\\4\end{pmatrix}=5^n\begin{pmatrix}100\\100\end{pmatrix}=5^n\mathbf{u}_0$$

(b) Similarly for $\mathbf{u}_0=\begin{pmatrix}100\\400\end{pmatrix}$ we have

$$c\begin{pmatrix}1\\1\end{pmatrix}+k\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}100\\400\end{pmatrix}\quad\Rightarrow\quad c=0,\ k=100$$

Using the formula, $\mathbf{A}^n\mathbf{w}=c\,\text{a}_1^n\mathbf{u}+k\,\text{a}_2^n\mathbf{v}$, of previous question we have

$$\mathbf{A}^n\begin{pmatrix}100\\400\end{pmatrix}=0\left(5^n\right)\begin{pmatrix}1\\1\end{pmatrix}+100\left(2^n\right)\begin{pmatrix}1\\4\end{pmatrix}=2^n\begin{pmatrix}100\\400\end{pmatrix}=2^n\mathbf{u}_0$$

(c) We need to write $\mathbf{u}_0=\begin{pmatrix}100\\200\end{pmatrix}$ as a linear combination of $\mathbf{u}=\begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\4\end{pmatrix}$:

$$c\mathbf{u}+k\mathbf{v}=c\begin{pmatrix}1\\1\end{pmatrix}+k\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}100\\250\end{pmatrix}\quad\Rightarrow\quad c=k=50$$

Using the formula $\mathbf{A}^n\mathbf{w}=c\,\text{a}_1^n\mathbf{u}+k\,\text{a}_2^n\mathbf{v}$ we have

$$\mathbf{A}^n\begin{pmatrix}100\\250\end{pmatrix}=50\left(5^n\right)\begin{pmatrix}1\\1\end{pmatrix}+50\left(2^n\right)\begin{pmatrix}1\\4\end{pmatrix}$$

$$=50\begin{pmatrix}5^n+2^n\\5^n+4\left(2^n\right)\end{pmatrix}$$

25. Let $\mathbf{A}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ then the eigenvectors of **A** are $\mathbf{u}=\begin{pmatrix}1\\-1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$. Evaluating

$$\mathbf{A}^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$$

The eigenvectors of $\mathbf{A}^2=\mathbf{I}$ are $\mathbf{u}'=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{v}'=\begin{pmatrix}0\\1\end{pmatrix}$. Hence the eigenvectors of $\mathbf{A}^2$ are different to the eigenvectors of **A**.

26. Using the characteristic equation $\det\left(\mathbf{M}-\text{a}\mathbf{I}\right)$ we have

$$\det(\mathbf{M}-\lambda\mathbf{I}) = \det\begin{pmatrix} \cos(\theta)-\lambda & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta)-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} \cos(\theta)-\lambda & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left[\left(\cos(\theta)-\lambda\right)\left(\cos(\theta)-\lambda\right)+\sin^2(\theta)\right]$$

$$= (1-\lambda)\left[\lambda^2-2\cos(\theta)\lambda+\cos^2(\theta)+\sin^2(\theta)\right]$$

$$= (1-\lambda)\left[\lambda^2-2\cos(\theta)\lambda+1\right]=0$$

One of the eigenvalues is $\lambda_1 = 1$. The others can be found by equating the quadratic to zero. We have

$$\lambda^2-2\cos(\theta)\lambda+1=0$$

Using the quadratic formula with $a=1,\ b=-2\cos(\theta),\ c=1$:

$$\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\cos(\theta)\pm\sqrt{4\cos^2(\theta)-4}}{2}$$

$$= \cos(\theta)\pm\sqrt{\cos^2(\theta)-1}$$

The real eigenvalues λ occur when $\cos(\theta)=\pm 1$ which implies $\theta = 0°,\ 180°$ but we are given that $0 < \theta < 180°$. Therefore there is *no* θ which produces a real eigenvalue from our quadratic. The only real eigenvalue we have is $\lambda_1 = 1$. Let **u** be the eigenvector belonging to $\lambda_1 = 1$:

$$(\mathbf{M}-\mathbf{I})\mathbf{u} = \begin{pmatrix} \cos(\theta)-1 & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta)-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding this out gives

$$\left.\begin{aligned} \left[\cos(\theta)-1\right]x+\sin(\theta)y=0 \\ -\sin(\theta)x+\left[\cos(\theta)-1\right]y=0 \end{aligned}\right\} \quad \Rightarrow x=y=0 \quad \text{and} \quad z=1$$

The eigenvector belonging to $\lambda_1 = 1$ is $\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The two other eigenvalues of **M** are complex numbers.

27. The eigenvalues and eigenvectors of matrix **T** are

$$\lambda_1 = 0.87,\ \mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 1,\ \mathbf{v} = \begin{pmatrix} 1 \\ 1.6 \end{pmatrix}$$

Since we need to find the matrix $\mathbf{T}^n$ so we have to create an invertible matrix **P**:

$$\mathbf{P} = \begin{pmatrix}\mathbf{u} & \mathbf{v}\end{pmatrix} = \begin{pmatrix}-1 & 1 \\ 1 & 1.6\end{pmatrix} \text{ and } \mathbf{P}^{-1} = \begin{pmatrix}-1 & 1 \\ 1 & 1.6\end{pmatrix}^{-1} = -\frac{1}{2.6}\begin{pmatrix}1.6 & -1 \\ -1 & -1\end{pmatrix} = \frac{1}{2.6}\begin{pmatrix}-1.6 & 1 \\ 1 & 1\end{pmatrix}$$

Using the following formula for evaluating the power of a matrix from chapter 7:

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

The diagonal matrix **D** is the eigenvalue matrix $\mathbf{D} = \begin{pmatrix}0.87 & 0 \\ 0 & 1\end{pmatrix}$. Hence

$$\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \begin{pmatrix}-1 & 1 \\ 1 & 1.6\end{pmatrix}\begin{pmatrix}0.87^n & 0 \\ 0 & 1^n\end{pmatrix}\frac{1}{2.6}\begin{pmatrix}-1.6 & 1 \\ 1 & 1\end{pmatrix}$$

As $n \to \infty$ we have $0.87^n \to 0$ (by the result of hint) and $1^n \to 1$. Hence as $n \to \infty$ we have

$$\begin{aligned}\mathbf{T}^n &= \frac{1}{2.6}\begin{pmatrix}-1 & 1 \\ 1 & 1.6\end{pmatrix}\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}-1.6 & 1 \\ 1 & 1\end{pmatrix} \\ &= \frac{1}{2.6}\begin{pmatrix}-1 & 1 \\ 1 & 1.6\end{pmatrix}\begin{pmatrix}0 & 0 \\ 1 & 1\end{pmatrix} \\ &= \frac{1}{2.6}\begin{pmatrix}1 & 1 \\ 1.6 & 1.6\end{pmatrix}\end{aligned}$$

The long term population distribution is given by

$$\mathbf{T}^n\mathbf{v}_0 = \frac{1}{2.6}\begin{pmatrix}1 & 1 \\ 1.6 & 1.6\end{pmatrix}\begin{pmatrix}56 \\ 6\end{pmatrix} = \frac{1}{2.6}\begin{pmatrix}62 \\ 99.2\end{pmatrix} = \begin{pmatrix}23.85 \\ 38.15\end{pmatrix}\begin{matrix}\text{U} \\ \text{R}\end{matrix}$$

The long term urban population will be about 24 million and the rural population about 38 million if the above trend continues to follow year on year.

28. Since **A** is a triangular matrix so the eigenvalues of **A** are $a_1 = 2,\ a_2 = 3$. The trace of a matrix is given by the sum of the eigenvalues:

> Proposition (7.9). Let **A** be any *n* by *n* matrix with eigenvalues $a_1,\ a_2,\ a_3,\ \cdots\ a_n$.
>
> (b) The trace of the matrix **A** is given by $tr(\mathbf{A}) = a_1 + a_2 + a_3 + \cdots + a_n$.

The eigenvalues of $\mathbf{A}^5$ are $a_1^{\,n} = 2^n,\ a_2^{\,n} = 3^n$ because

> Proposition (7.8). Let **A** have an eigenvector **u** belonging to eigenvalue a. Then
>
> (a) If *m* is a natural number then a^m is an eigenvalue of the matrix $\mathbf{A}^m$ with the *same* eigenvector **u**.

By (7.9) we have $tr(\mathbf{A}^n) = 2^n + 3^n$.

29. $\mathbf{A} - a\mathbf{I}$ equal to

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 0 & 2 & -1\\ 0 & 1-\lambda & 4 & -2\\ 2 & -1 & -\lambda & 1\\ 2 & -1 & -1 & 2-\lambda\end{pmatrix}$$

To evaluate the determinant of a 4 by 4 matrix can be tedious task so we apply some row operations to reduce the number of calculations we have to carry out in order to work out the determinant. Subtracting the bottom two rows gives

$$\begin{matrix}R_1\\ R_2\\ R_3\\ R_4^*=R_4-R_3\end{matrix}\quad\begin{pmatrix}1-\lambda & 0 & 2 & -1\\ 0 & 1-\lambda & 4 & -2\\ 2 & -1 & -\lambda & 1\\ 0 & 0 & \lambda-1 & 1-\lambda\end{pmatrix}$$

Carrying out the row operation $2R_1-R_2$ gives

$$\begin{matrix}2R_1-R_2\\ R_2\\ R_3\\ R_4^*\end{matrix}\quad\begin{pmatrix}2-2\lambda & \lambda-1 & 0 & 0\\ 0 & 1-\lambda & 4 & -2\\ 2 & -1 & -\lambda & 1\\ 0 & 0 & \lambda-1 & 1-\lambda\end{pmatrix}$$

Evaluating the determinant of this last matrix by expanding along the bottom row gives:

$$\det\begin{pmatrix}2-2\lambda & \lambda-1 & 0 & 0\\ 0 & 1-\lambda & 4 & -2\\ 2 & -1 & -\lambda & 1\\ 0 & 0 & \lambda-1 & 1-\lambda\end{pmatrix}=-(\lambda-1)\det\begin{pmatrix}2-2\lambda & \lambda-1 & 0\\ 0 & 1-\lambda & -2\\ 2 & -1 & 1\end{pmatrix}+(1-\lambda)\det\begin{pmatrix}2-2\lambda & \lambda-1 & 0\\ 0 & 1-\lambda & 4\\ 2 & -1 & -\lambda\end{pmatrix}$$

$$=(1-\lambda)\left[(2-2\lambda)\det\begin{pmatrix}1-\lambda & -2\\ -1 & 1\end{pmatrix}-(\lambda-1)\det\begin{pmatrix}0 & -2\\ 2 & 1\end{pmatrix}\right]$$

$$+(1-\lambda)\left[(2-2\lambda)\det\begin{pmatrix}1-\lambda & 4\\ -1 & -\lambda\end{pmatrix}-(\lambda-1)\det\begin{pmatrix}0 & 4\\ 2 & -\lambda\end{pmatrix}\right]$$

$$=(1-\lambda)^2\left[2\left[(1-\lambda)-2\right]+4\right]+(1-\lambda)^2\left[2(-\lambda)(1-\lambda)+8-8\right]$$

$$=(1-\lambda)^2\left\{\left[2-2\lambda\right]+\left[2\lambda^2-2\lambda\right]\right\}$$

$$=(1-\lambda)^2\left\{2\lambda^2-4\lambda+2\right\}$$

$$=2(1-\lambda)^2\left\{\lambda^2-2\lambda+1\right\}=2(1-\lambda)^2(\lambda-1)^2$$

Hence $\lambda_{1,\,2,\,3,\,4}=1$. Our eigenvectors belonging to this eigenvalue are given by

$$\begin{pmatrix}0 & 0 & 2 & -1\\ 0 & 0 & 4 & -2\\ 2 & -1 & -1 & 1\\ 2 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\\ w\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

Writing the augmented matrix and solving gives

$$\begin{array}{r} \\ R_1 \\ R_2-2R_1 \\ R_3 \\ R_4-R_3 \end{array}\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \qquad (*)$$

From the top row we have $2z = w$. Let $w = 2$ then $z = 1$. From the third row we have

$$2x - y - z + w = 0$$

$$2x - y - 1 + 2 = 0 \quad \Rightarrow \quad 2x = y - 1$$

Let $y = 3$ then $x = 1$. One vector belonging to the eigenvalue $a_{1,\ 2,\ 3,\ 4} = 1$ is

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}$$

Another *independent* eigenvector can be found by using the above result $2z = w$ with $w = 0$ which gives $z = 0$. Putting these into the third row of (*) gives

$$2x - y = 0 \ \Rightarrow y = 2x$$

Let $x = 1$ then $y = 2$. This eigenvector is

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

The two linearly eigenvectors (we have 2 non-zero rows in (*)) belonging to $a_{1,\ 2,\ 3,\ 4} = 1$ are

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

30. (i) From the definition of the norm of a vector :

(4.3)	$\lVert\mathbf{u}\rVert = \sqrt{\langle \mathbf{u},\ \mathbf{u} \rangle}$

Using $\langle \mathbf{u},\ \mathbf{u} \rangle = \mathbf{u}^T\mathbf{u}$ we have

$$\lVert\mathbf{Ax}\rVert^2 = (\mathbf{Ax})^T(\mathbf{Ax}) = \left(\mathbf{x}^T\mathbf{A}^T\right)\mathbf{Ax} = \mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x}$$

(ii) *How do we show a matrix is symmetric?*

A square matrix **A** is a symmetric matrix if $\mathbf{A}^T = \mathbf{A}$. Just need to check this property:

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T \qquad \left[\text{Because Theorem (1.9)(d) } \left(\mathbf{XY}\right)^T = \mathbf{Y}^T\mathbf{X}^T\right]$$

$$= \mathbf{A}^T\mathbf{A} \qquad \left[\text{Because Theorem (1.9)(a) } \left(\mathbf{X}^T\right)^T = \mathbf{X}\right]$$

Hence $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

(iii) First we find $\mathbf{A}^T\mathbf{A}$ for the given matrix:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{pmatrix}$$

Using Matlab to find the eigenvalues of $\mathbf{A}^T\mathbf{A}$:

$$a_1 = 0,\ a_2 = 0.5973 \text{ and } a_3 = 90.4027$$

The largest eigenvalue is $a_3 = 90.4027$ so $\|\mathbf{A}\| = \sqrt{90.4027} = 9.508$ (3dp)

(iv) The command norm(A) gives the value found in part(iii), 9.508. Hence the command norm(A) in Matlab evaluates the norm of a matrix as defined in the question.

31. No because if we evaluate the vector $\mathbf{Ax}$ then this clearly is not a multiple of **x**:

$$\mathbf{Ax} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 26 \\ 42 \\ 58 \end{pmatrix} \neq k\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

32. (i) For SVD we need to first find $\mathbf{A}^T\mathbf{A}$:

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}^T\mathbf{A}$ are

$$a_1 = 6,\ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } a_2 = 1,\ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

The normalized eigenvectors are $\widehat{\mathbf{v}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\widehat{\mathbf{v}}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

The singular values, $\pm_1$ and $\pm_2$, are the square roots of the eigenvalues:

$$\pm_1 = \sqrt{a_1} = \sqrt{6}, \text{ and } \pm_2 = \sqrt{a_2} = 1$$

The orthogonal matrix $\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$ where $\mathbf{u}_1 = \frac{1}{\pm_1}\mathbf{A}\widehat{\mathbf{v}_1}$ and $\mathbf{u}_2 = \frac{1}{\pm_2}\mathbf{A}\widehat{\mathbf{v}_2}$:

$$\mathbf{u}_1 = \frac{1}{\pm_1}\mathbf{A}\widehat{\mathbf{v}_1} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\pm_2}\mathbf{A}\widehat{\mathbf{v}_2} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Because $\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$ is an orthogonal matrix so vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ must be orthonormal vectors.

Our diagonal matrix $\mathbf{D} = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{U} = \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} \\ 1/\sqrt{30} & 2/\sqrt{5} \\ 5/\sqrt{30} & 0 \end{pmatrix}$ and $\mathbf{V} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$:

$$\mathbf{A} = \mathbf{UDV}^T = \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} \\ 1/\sqrt{30} & 2/\sqrt{5} \\ 5/\sqrt{30} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\right]^T$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} \\ 1/\sqrt{30} & 2/\sqrt{5} \\ 5/\sqrt{30} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

(ii) For the *full* svd we need to find a unit vector $\mathbf{u}_3$ which is orthogonal to

$$\mathbf{u}_1 = \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \text{ and } \mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Ignore the scalars because we will need to normalize $\mathbf{u}_3$:

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Solving the equations

$$\left.\begin{aligned} 2x + y + 5z &= 0 \\ -x + 2y \quad\quad &= 0 \end{aligned}\right\} \Rightarrow \quad x = 2,\ y = 1,\ z = -1$$

Hence $\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

For the *full* SVD we have $\mathbf{U}' = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{pmatrix} = \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix}$ and

$\mathbf{D}' = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ so

$$\mathbf{A} = \mathbf{U}'\mathbf{D}'\mathbf{V}^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

33. No because matrix **A** has eigenvalues 1, 4 and 6 but matrix **B** has eigenvalues 1, 1 and 1. Remember by the following proposition, similar matrices must have the same eigenvalues:

> Proposition (7.14). Let **A** and **B** be similar matrices. The eigenvalues of these matrices are identical.

34. (i) Remember the trace of a matrix gives the sum of the eigenvalues because:

> Proposition (7.9). Let **A** be any *n* by *n* matrix with eigenvalues $a_1, a_2, a_3, \cdots a_n$. We have:
>
> (a) The determinant of the matrix **A** is given by $\det(\mathbf{A}) = a_1 \times a_2 \times a_3 \times \cdots \times a_n$.
>
> (b) The trace of the matrix **A** is given by $tr(\mathbf{A}) = a_1 + a_2 + a_3 + \cdots + a_n$.

We are given $a_1 = -4$ and $a_2 = 3$ are two eigenvalues of matrix **A**. The trace of the matrix is sum of the entries on the leading diagonal:

$$tr(\mathbf{A}) = tr\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} = 5 - 1 - 2 = 2$$

By using Proposition (7.9) part (b) we have

$$a_1 + a_2 + a_3 = -4 + 3 + a_3 = 2 \quad \Leftrightarrow \quad a_3 = 3$$

Our third eigenvector is $a_3 = 3$.

(ii) Since none of the eigenvalues are equal to zero so our given matrix **A** is invertible.

(iii) To find the determinant we use the above Proposition (7.9) part (a):

$$\det(\mathbf{A}) = a_1 \times a_2 \times a_3 = -4 \times 3 \times 3 = -36$$

35. We are given that $\mathbf{A}=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$. In order to find the diagonalizing matrix we need to find the eigenvalues and eigenvectors of this matrix. The eigenvalues of **A** are given by $\det(\mathbf{A}-a\mathbf{I})=0$. We need to carry out row operations in order to evaluate the eigenvalues because finding the determinant of a 4 by 4 with no zeros can be tedious task.

$$\begin{matrix}R_1\\ R_2\\ R_3\\ R_4\end{matrix}\begin{pmatrix}1-a & 1 & 1 & 1\\ 1 & 1-a & -1 & -1\\ 1 & -1 & 1-a & -1\\ 1 & -1 & -1 & 1-a\end{pmatrix}$$

Carrying out the following row operations

$$\begin{matrix}R_1+R_2\\ R_2\\ R_3-R_2\\ R_4+R_1\end{matrix}\begin{pmatrix}2-a & 2-a & 0 & 0\\ 1 & 1-a & -1 & -1\\ 0 & a-2 & 2-a & 0\\ 2-a & 0 & 0 & 2-a\end{pmatrix}=\mathbf{B}$$

None of these operations make any difference to the determinant.
Evaluating the determinant of this matrix **B** we have

$$\det(\mathbf{B})=\det\begin{pmatrix}2-a & 2-a & 0 & 0\\ 1 & 1-a & -1 & -1\\ 0 & a-2 & 2-a & 0\\ 2-a & 0 & 0 & 2-a\end{pmatrix}$$

$$=(2-a)\det\begin{pmatrix}1-a & -1 & -1\\ a-2 & 2-a & 0\\ 0 & 0 & 2-a\end{pmatrix}-(2-a)\det\begin{pmatrix}1 & -1 & -1\\ 0 & 2-a & 0\\ 2-a & 0 & 2-a\end{pmatrix}$$

$$=(2-a)(2-a)\det\begin{pmatrix}1-a & -1\\ a-2 & 2-a\end{pmatrix}-(2-a)(2-a)\det\begin{pmatrix}1 & -1\\ 2-a & 2-a\end{pmatrix}$$

$$=(2-a)^2\left[(1-a)(2-a)+(a-2)-\left((2-a)+(2-a)\right)\right]$$

$$=(2-a)^2\left[(1-a)(2-a)-(2-a)-2(2-a)\right]$$

$$=(2-a)^3\left[(1-a)-1-2\right]=(2-a)^3(-a-2)=0$$

Our eigenvalues are $a_{1,\,2,\,3}=2,\ a_4=-2$.

The orthogonal matrix **Q** is constructed by the eigenvectors.

Let **u**, **v** and **w** be the eigenvectors belonging to $a_{1,\,2,\,3}=2$.

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this by using row operations:

$$\begin{array}{r} \\ R_1 \\ R_2+R_1 \\ R_3+R_1 \\ R_4+R_1 \end{array}\left(\begin{array}{cccc|c} x & y & z & w & \\ -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the top row we have

$$x = y + z + w \qquad (*)$$

where *y*, *z* and *w* free variables. Let $y = z = w = 1$. Then $x = 3$.

One eigenvector belonging to $a_{1,\ 2,\ 3} = 2$ is

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We need 2 more vectors which are orthogonal to this vector **u**.

Let $y = -2,\ z = w = 1$ then $x = y + z + w = -2 + 1 + 1 = 0$. Hence another eigenvector is

$$\mathbf{v} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

We need our third eigenvector $\mathbf{w} = \begin{pmatrix} x & y & z & w \end{pmatrix}^T$ to be orthogonal to both **u** and **v**.

This means we need

$$\mathbf{u}\cdot\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} x \underbrace{=}_{\text{By }(*)} y+z+w \\ y \\ z \\ w \end{pmatrix} = 3(y+z+w) + y + z + w = 0,\ \ \mathbf{v}\cdot\mathbf{w} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = -2y + z + w = 0$$

From the first equation we have

$$4(y+z+w) = 0 \ \Rightarrow \ y + z + w = 0$$

The second equation is $-2y + z + w = 0$. Equating these we have

$$y + z + w = -2y + z + w = 0 \ \Rightarrow \ y = 0$$

Substituting this $y = 0$ into $y + z + w = 0$ gives $z = -w$. Let $w = 1$ then $z = -1$ and $x = 0$. Hence $\mathbf{w} = \begin{pmatrix} x & y & z & w \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & -1 & 1 \end{pmatrix}^T$.

We need to find the eigenvector belonging to $a_4 = -2$. Let **x** belong to the eigenvalue $a_4 = -2$:

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Collecting our eigenvectors we have

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

All these vectors are orthogonal to each other. For an orthogonal matrix **Q** the columns (or rows) need to be orthonormal, that is orthogonal and normalized. Normalizing the eigenvectors gives

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{12}}\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \hat{\mathbf{v}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \ \hat{\mathbf{w}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \hat{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Hence our orthogonal matrix $\mathbf{Q} = \begin{pmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{w}} & \hat{\mathbf{x}} \end{pmatrix}$.

36. *Proof*.

We have $\mathbf{Av} = a\mathbf{v}$ where **v** is the eigenvector belonging to the eigenvalue a.
Consider the matrix $k\mathbf{A}$:

$$k\mathbf{Av} = k\left(a\mathbf{v}\right) = \left(ka\right)\mathbf{v}$$

We have $\underbrace{\left(k\mathbf{A}\right)}_{\text{Matrix}}\mathbf{v} = \underbrace{\left(ka\right)}_{\text{Scalar}}\mathbf{v}$ which means that the matrix $k\mathbf{A}$ has eigenvalue ka with eigenvector **v.**

■

Since the $k\mathbf{A}$ and **A** have the same eigenvectors so the eigenvector matrix **Q** is identical to the one found in the previous question for $\frac{1}{2}\mathbf{A}$. Of course the eigenvalues are different, they are actually $a_{1,2,3} = \frac{1}{2}(2) = 1, \ a_4 = \frac{1}{2}(-2) = -1$.

37. (a) Using the result of the previous question, $k\mathbf{A}$ has eigenvalue ka, we have the eigenvalues of 3**A** are

$$3 \times 1 = 3, \ 6, \ 9, \ 12 \text{ and } 15$$

(b) Using the result of question 16 Exercise 7(b):

The eigenvalues of the transposed matrix, $\mathbf{A}^T$, are exactly the eigenvalues of the matrix **A**.

We have the eigenvalues of $\mathbf{A}^T$ are 1, 2, 3 , 4 and 5.

(c) Using the following result of chapter 7:

Proposition (7.8). Let **A** be a square matrix with eigenvector **u** belonging to eigenvalue a .

(b) If the matrix **A** is invertible (has an inverse) then the eigenvalue of the inverse matrix $\mathbf{A}^{-1}$ is $\frac{1}{a} = a^{-1}$ with the *same* eigenvector **u**.

Hence the eigenvalues of $\mathbf{A}^{-1}$ are

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \text{ and } \frac{1}{5}$$

38. *Proof*.

Since matrix **A** is orthogonal so by question 21 of Exercise 6(c):

$$\det(\mathbf{A}) = \pm 1$$

This means the matrix **A** is invertible and the eigenvalues cannot equal zero. By

Proposition (7.9). Let **A** be any *n* by *n* matrix with eigenvalues $a_1, a_2, a_3, \cdots a_n$. We have:

(a) The determinant of the matrix **A** is given by $\det(\mathbf{A}) = a_1 \times a_2 \times a_3 \times \cdots \times a_n$.

This means $\det(\mathbf{A}) = a_1 \times a_2 = \pm 1$. Let a_1 be one of the eigenvalues of matrix **A**. Then in order to satisfy $a_1 \times a_2 = \pm 1$ we must have $a_2 = \pm \frac{1}{a_1}$.

■

39. No because it may have the zero eigenvalue amongst its' distinct eigenvalues so by:

Proposition (7.7). A square matrix **A** is invertible (has an inverse) $\Leftrightarrow$ $a = 0$ is *not* an eigenvalue of the matrix **A**.

Matrix **A** may be non-invertible.

40. Let **u** be the eigenvector belonging to the eigenvalue λ of the given matrix **A**. Then by the definition of eigenvalue:

(7.1)	$\mathbf{Au} = \lambda\mathbf{u}$

Consider the matrix $\mathbf{A} + n\mathbf{I}$. Multiplying this by the eigenvector **u** gives

$$(\mathbf{A} + n\mathbf{I})\mathbf{u} = \mathbf{Au} + n\mathbf{u} = \lambda\mathbf{u} + n\mathbf{u} = (\lambda + n)\mathbf{u}$$

The eigenvalue of $\mathbf{A} + n\mathbf{I}$ is $\lambda + n$.

41. Taking the half out we have $\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Finding the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1$$

$$= \lambda^2 - 2\lambda + 1 - 1 = \lambda(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \ \lambda_2 = 2$$

Using the result of question 36 which says the eigenvalues of the matrix $k\mathbf{A}$ are $k\lambda$. Let t_1 and t_2 be the eigenvalues of the given matrix then

$$t_1 = \frac{1}{2}(0) = 0 \ \text{ and } \ t_2 = \frac{1}{2}(2) = 1$$

We have that both matrices $\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ have the same eigenvector therefore we find the eigenvectors of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Let **u** be the eigenvector belonging to $\lambda_1 = 0$ then

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad [\text{Because } \ x = 1, \ y = -1]$$

Similarly let **v** be the eigenvector belonging to $\lambda_2 = 2$ then

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad [\text{Because } \ x = y = 1]$$

The invertible matrix $\mathbf{P} = (\mathbf{u} \quad \mathbf{v}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and the inverse of this is

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Since the eigenvalues are $t_1 = 0$ and $t_2 = 1$ so the eigenvalue matrix $\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We determine $\mathbf{A}^5$ by using the following of chapter 7:

$$\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$$

Substituting $\mathbf{P}=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ and $\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ into this formula

$$\begin{aligned}\mathbf{A}^m &= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^m \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \mathbf{A}\end{aligned}$$

Hence $\mathbf{A}^m=\mathbf{A}$.

42. Let $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix}$ be a stochastic matrix and vector respectively. We have

$$a+c=1,\ b+d=1 \ \text{ and } \ x+y=1 \qquad (*)$$

Applying the matrix **A** to the vector **x** gives

$$\mathbf{Ax}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}ax+by\\ cx+dy\end{pmatrix}$$

Adding the two entries of **Ax** and using (*) gives

$$ax+by+cx+dy=\underbrace{(a+c)}_{=1}x+\underbrace{(b+d)}_{=1}y=\underbrace{x+y}_{=1}=1$$

Hence **Ax** is a stochastic vector because the sum of the entries is 1 and none of the entries is negative.

43. Evaluating $\mathbf{Au}$ and $\mathbf{Av}$:

$$\mathbf{Au}=\begin{pmatrix}1-a & b\\ a & 1-b\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}1-a-b\\ a+b-1\end{pmatrix}=\begin{pmatrix}1-a-b\\ -(1-a-b)\end{pmatrix}=(1-a-b)\begin{pmatrix}1\\ -1\end{pmatrix}$$

The eigenvalue corresponding to eigenvector **u** is $\mathrm{a}_1=1-a-b$. Similarly

$$\mathbf{Av}=\begin{pmatrix}1-a & b\\ a & 1-b\end{pmatrix}\begin{pmatrix}b\\ a\end{pmatrix}=\begin{pmatrix}b-ab+ba\\ ab+a-ab\end{pmatrix}=\begin{pmatrix}b\\ a\end{pmatrix}=\mathbf{v}$$

Hence the eigenvalue corresponding to **v** is $\mathrm{a}_2=1$.

44. The eigenvalues of the $\mathbf{A}=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}$ are $\mathrm{a}_1=-1$, $\mathrm{a}_2=1$ and $\mathrm{a}_3=3$ with the corresponding eigenvectors;

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$ respectively.

The eigenvector matrix $\mathbf{P} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. The inverse of this matrix $\mathbf{P}^{-1}$

can be found by applying row operations:

$$\begin{pmatrix} \mathbf{P} & \mathbf{I} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

Carrying out the following row operation:

$$\begin{matrix} R_1 \\ R_2 \\ R_3^{\dagger} = R_3 + R_1 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

Executing the row operation:

$$\begin{matrix} 2R_1 - R_3^{\dagger} \\ R_2 \\ R_3^{\dagger} \end{matrix} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

Multiplying the top and bottom rows by ½ and taking out the common factor ½ gives

$$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We use the formula of chapter 7:

$$\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$$

*What is the eigenvalue matrix **D** equal to?*

$$\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Substituting $\mathbf{P}=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&3\end{pmatrix}$ and $\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}1&0&-1\\0&2&0\\1&0&1\end{pmatrix}$ into

$\mathbf{A}^{10}=\mathbf{PD}^{10}\mathbf{P}^{-1}$ is:

$$\mathbf{A}^{10}=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&3\end{pmatrix}^{10}\frac{1}{2}\begin{pmatrix}1&0&-1\\0&2&0\\1&0&1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&59049\end{pmatrix}\begin{pmatrix}1&0&-1\\0&2&0\\1&0&1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&-1\\0&2&0\\59049&0&59049\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}59050&0&59048\\0&2&0\\59048&0&59050\end{pmatrix}=\begin{pmatrix}29525&0&29524\\0&1&0\\29524&0&29525\end{pmatrix}$$

Substituting $\mathbf{A}^{10}=\begin{pmatrix}29525&0&29524\\0&1&0\\29524&0&29525\end{pmatrix}$ and $\mathbf{v}_0=\begin{pmatrix}1\\1\\1\end{pmatrix}$ into $\mathbf{v}_{10}=\mathbf{A}^{10}\mathbf{v}_0$ gives

$$\mathbf{v}_{10}=\begin{pmatrix}29525&0&29524\\0&1&0\\29524&0&29525\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}59049\\1\\59049\end{pmatrix}$$

45. (a) Let $\mathbf{A}=\begin{pmatrix}3&0&4\\0&5&0\\4&0&3\end{pmatrix}$. The eigenvalues and corresponding eigenvectors of matrix **A** are

$$a_1=-1,\mathbf{u}=\begin{pmatrix}1\\0\\-1\end{pmatrix},\ a_2=5,\ \mathbf{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}\ \text{and}\ a_3=7,\ \mathbf{w}=\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad (*)$$

Since we have distinct eigenvalues so our eigenvectors are linearly independent. We are given that the general solution is

$$\mathbf{x}=c_1\mathbf{u}_1e^{a_1 t}+c_2\mathbf{u}_2e^{a_2 t}+\cdots+c_n\mathbf{u}_ne^{a_n t}$$

Substituting the above eigenvalues and eigenvectors of (*) into this formula gives:

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{5t} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{7t}$$

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The eigenvalues and corresponding eigenvectors of matrix **A** are

$$\text{a}_1 = 0, \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \ \text{a}_2 = 0, \ \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \text{a}_3 = 3, \ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (**)$$

We do not have distinct eigenvalues so we need to check that these eigenvectors are linearly independent. We can use the determinant to check this:

$$\det\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

$$= -2\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2[2] + [-2] = -6$$

Since the determinant is *non-zero* so the eigenvectors are linearly independent. Substituting the above eigenvalues and eigenvectors of (**) into the given formula:

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{0} + c_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{0} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} = c_1\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} \quad \left[\text{Because } e^{0} = 1\right]$$

46. For the given matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ the eigenvalues and eigenvectors are given by:

$$\text{a}_1 = 2, \ \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \text{a}_2 = -3, \ \mathbf{v} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

The eigenvector matrix $\mathbf{P} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}$ and the eigenvalue matrix

$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$. The inverse of **P** is $\mathbf{P}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}$.

Substituting this into the given formula $e^{t\mathbf{A}} = \mathbf{P}\begin{pmatrix} e^{\text{a}_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\text{a}_n t} \end{pmatrix}\mathbf{P}^{-1}$ yields

$$e^{t\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}} = \begin{pmatrix}1 & -1\\ 1 & 4\end{pmatrix}\begin{pmatrix}e^{2t} & 0\\ 0 & e^{-3t}\end{pmatrix}\frac{1}{5}\begin{pmatrix}4 & 1\\ -1 & 1\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}1 & -1\\ 1 & 4\end{pmatrix}\begin{pmatrix}4e^{2t} & e^{2t}\\ -e^{-3t} & e^{-3t}\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}4e^{2t}+e^{-3t} & e^{2t}-e^{-3t}\\ 4e^{2t}-4e^{-3t} & e^{2t}+4e^{-3t}\end{pmatrix}$$

47. Substituting the result of the above question

$$e^{t\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}} = \frac{1}{5}\begin{pmatrix}4e^{2t}+e^{-3t} & e^{2t}-e^{-3t}\\ 4e^{2t}-4e^{-3t} & e^{2t}+4e^{-3t}\end{pmatrix}$$

And $\mathbf{x}_0 = \begin{pmatrix}1\\ 2\end{pmatrix}$ into the given formula $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0$ yields

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 = \frac{1}{5}\begin{pmatrix}4e^{2t}+e^{-3t} & e^{2t}-e^{-3t}\\ 4e^{2t}-4e^{-3t} & e^{2t}+4e^{-3t}\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}6e^{2t}-e^{-3t}\\ 6e^{2t}+4e^{-3t}\end{pmatrix}$$

48. *Proof*.

$(\Rightarrow)$. We assume matrices **A** and **B** are similar so there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$ because:

Definition (7.12). A square matrix **B** is **similar** to a matrix **A** if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$.

We are also told that **A** is diagonalizable, so there is an invertible matrix **Q** such that

$$\mathbf{Q}^{-1}\mathbf{AQ} = \mathbf{D} \qquad (\dagger)$$

where **D** is a diagonal matrix.

From $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$ we have

$$\mathbf{A} = \mathbf{PBP}^{-1} \qquad (*)$$

Substituting this into (†) gives

$$\mathbf{Q}^{-1}\left(\mathbf{PBP}^{-1}\right)\mathbf{Q} = \mathbf{D}$$

Using the associative rule of matrices we have

$$\left(\mathbf{Q}^{-1}\mathbf{P}\right)\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{Q}\right) = \left(\mathbf{Q}^{-1}\left(\mathbf{P}^{-1}\right)^{-1}\right)\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{Q}\right)$$

$$= \left(\mathbf{P}^{-1}\mathbf{Q}\right)^{-1}\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{Q}\right) = \mathbf{D}$$

We have $\left(\mathbf{P}^{-1}\mathbf{Q}\right)^{-1}\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{Q}\right)=\mathbf{D}$ where **D** is a diagonal matrix so by the above definition (7.12) we conclude matrix **B** is similar to diagonal matrix which means it is diagonalizable. This completes our proof for this half $(\Rightarrow)$.

$(\Leftarrow)$. This time we assume matrices **A** and **B** are diagonalizable with the same diagonal matrix **D**. This means there are invertible matrices **P** and **Q** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\mathbf{D} \quad \text{and} \quad \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}=\mathbf{D}$$

Equating these we have $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}$. Left multiplying this by **Q** and right multiplying this by $\mathbf{Q}^{-1}$ gives

$$\mathbf{Q}\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{Q}^{-1}=\mathbf{Q}\left(\mathbf{Q}^{-1}\mathbf{B}\mathbf{Q}\right)\mathbf{Q}^{-1}$$

$$\left(\mathbf{Q}\mathbf{P}^{-1}\right)\mathbf{A}\left(\mathbf{P}\mathbf{Q}^{-1}\right)=\left(\mathbf{Q}\mathbf{Q}^{-1}\right)\mathbf{B}\left(\mathbf{Q}\mathbf{Q}^{-1}\right)$$

$$\left(\mathbf{P}\mathbf{Q}^{-1}\right)^{-1}\mathbf{A}\left(\mathbf{P}\mathbf{Q}^{-1}\right)=\mathbf{B}$$

Since $\mathbf{P}\mathbf{Q}^{-1}$ is invertible so matrices **A** and **B** are similar.

■

49. Both matrices $\mathbf{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1&-3\\1&3\end{pmatrix}$ have eigenvalues $\lambda_1=0$ and $\lambda_2=2$. By

> Proposition (7.18). If an *n* by *n* matrix **A** has *n distinct* eigenvalues then the matrix **A** is diagonalizable.

Matrices **A** and **B** are diagonalizable with the same eigenvalue matrix **D** because they both have the same eigenvalues. By result of previous question matrices **A** and **B** are similar.

50. We need to show that $\mathbf{A}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&1&3\\0&2&4\\0&0&2\end{pmatrix}$ are *not* similar.

Both matrices $\mathbf{A}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&1&3\\0&2&4\\0&0&2\end{pmatrix}$ have the same eigenvalues

$$\lambda_{1,\,2,\,3}=2$$

Clearly **A** is diagonalisable because **A** is a diagonal matrix so our invertible matrix is **I**.

Matrix **B** has only one linearly independent eigenvector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ so we cannot form an invertible 3 by 3 eigenvector matrix **P** such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is a diagonal matrix which implies that matrix **B** is not diagonalizable. By the result of question 48 we conclude that matrices **A** and **B** are *not* similar even though they have the same eigenvalues.

51. In each case we have $-\lambda$ on the leading diagonal because we examine $\mathbf{A}-\lambda\mathbf{I}$.

(a) We have

$$\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\begin{pmatrix}-\lambda & 1\\ 2 & -\lambda\end{pmatrix}=\lambda^2-\left(2\times 1\right)=0 \quad\Rightarrow\quad \lambda_{1,\,2}=\pm\sqrt{2\times 1}=\pm\sqrt{2}$$

(b) Similarly we have

$$\det\left(\mathbf{B}-\lambda\mathbf{I}\right)=\begin{pmatrix}-\lambda & 0 & 0 & 1\\ 0 & -\lambda & 2 & 0\\ 0 & 3 & -\lambda & 0\\ 4 & 0 & 0 & -\lambda\end{pmatrix}=-\lambda\det\begin{pmatrix}-\lambda & 2 & 0\\ 3 & -\lambda & 0\\ 0 & 0 & -\lambda\end{pmatrix}-\det\begin{pmatrix}0 & -\lambda & 2\\ 0 & 3 & -\lambda\\ 4 & 0 & 0\end{pmatrix}$$

$$=-\lambda\left[-\lambda\det\begin{pmatrix}-\lambda & 2\\ 3 & -\lambda\end{pmatrix}\right]-4\det\begin{pmatrix}-\lambda & 2\\ 3 & -\lambda\end{pmatrix}$$

$$=-\lambda\left[-\lambda\left(\lambda^2-\left(3\times 2\right)\right)\right]-4\left(\lambda^2-\left(3\times 2\right)\right)$$

$$=\left(\lambda^2-\left(3\times 2\right)\right)\left(\lambda^2-4\right)=0$$

We have $\lambda_{1,\,2}=\pm\sqrt{3\times 2},\ \lambda_{3,\,4}=\pm\sqrt{4}=\pm 2$

(c) Repeating the above process for the 6 by 6 matrix we have

$$\det\left(\mathbf{C}-\lambda\mathbf{I}\right)=\det\begin{pmatrix}-\lambda & 0 & 0 & 0 & 0 & 1\\ 0 & -\lambda & 0 & 0 & 2 & 0\\ 0 & 0 & -\lambda & 3 & 0 & 0\\ 0 & 0 & 4 & -\lambda & 0 & 0\\ 0 & 5 & 0 & 0 & -\lambda & 0\\ 6 & 0 & 0 & 0 & 0 & -\lambda\end{pmatrix}$$

$$=-\lambda\det\begin{pmatrix}-\lambda & 0 & 0 & 2 & 0\\ 0 & -\lambda & 3 & 0 & 0\\ 0 & 4 & -\lambda & 0 & 0\\ 5 & 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & 0 & -\lambda\end{pmatrix}-\det\begin{pmatrix}0 & -\lambda & 0 & 0 & 2\\ 0 & 0 & -\lambda & 3 & 0\\ 0 & 0 & 4 & -\lambda & 0\\ 0 & 5 & 0 & 0 & -\lambda\\ 6 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$=-\lambda\left(-\lambda\right)\det\begin{pmatrix}-\lambda & 0 & 0 & 2\\ 0 & -\lambda & 3 & 0\\ 0 & 4 & -\lambda & 0\\ 5 & 0 & 0 & -\lambda\end{pmatrix}-6\det\begin{pmatrix}-\lambda & 0 & 0 & 2\\ 0 & -\lambda & 3 & 0\\ 0 & 4 & -\lambda & 0\\ 5 & 0 & 0 & -\lambda\end{pmatrix}$$

$$\underbrace{=}_{\text{Factorizing}}\left[\lambda^2-6\right]\det\begin{pmatrix}-\lambda & 0 & 0 & 2\\ 0 & -\lambda & 3 & 0\\ 0 & 4 & -\lambda & 0\\ 5 & 0 & 0 & -\lambda\end{pmatrix}\qquad(\dagger)$$

The determinant of the last 4 by 4 matrix is given by

$$\det\begin{pmatrix} -\lambda & 0 & 0 & 2 \\ 0 & -\lambda & 3 & 0 \\ 0 & 4 & -\lambda & 0 \\ 5 & 0 & 0 & -\lambda \end{pmatrix} = -\lambda \det\begin{pmatrix} -\lambda & 3 & 0 \\ 4 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} - 2\det\begin{pmatrix} 0 & -\lambda & 3 \\ 0 & 4 & -\lambda \\ 5 & 0 & 0 \end{pmatrix}$$

$$= (-\lambda)(-\lambda)\det\begin{pmatrix} -\lambda & 3 \\ 4 & -\lambda \end{pmatrix} - (2\times 5)\det\begin{pmatrix} -\lambda & 3 \\ 4 & -\lambda \end{pmatrix}$$

$$= \left[\lambda^2 - (2\times 5)\right]\det\begin{pmatrix} -\lambda & 3 \\ 4 & -\lambda \end{pmatrix} = \left[\lambda^2 - (2\times 5)\right]\left[\lambda^2 - (3\times 4)\right]$$

Putting this into $(\dagger)$ and equating to zero gives

$$\left[\lambda^2 - 6\right]\left[\lambda^2 - (2\times 5)\right]\left[\lambda^2 - (3\times 4)\right] = 0 \quad \Rightarrow \quad \lambda_{1,\,2} = \pm\sqrt{6},\ \lambda_{3,\,4} = \pm\sqrt{2\times 5},\ \lambda_{5,\,6} = \pm\sqrt{3\times 4}$$

Prediction is that if $\mu_1,\ \mu_2,\ \cdots,\ \mu_{2n}$ are the entries on the secondary diagonal and the matrix has zeros entries elsewhere then the eigenvalues are given by

$$\lambda_{1,\,2} = \pm\sqrt{\mu_1\times\mu_{2n}},\ \lambda_{3,\,4} = \pm\sqrt{\mu_2\times\mu_{2n-1}},\ \lambda_{5,\,6} = \pm\sqrt{\mu_3\times\mu_{2n-2}},\ \cdots,\ \lambda_{2n-1,\,2n} = \sqrt{\mu_n\mu_{n+1}}$$