Complete Solutions to Problems of Chapter 5 (Only available to Tutors)

1. We need to check the following for linear transformation:

> Definition (5.2). A transformation $T:\ V \to W$ is called a **linear transformation** $\Leftrightarrow$ for *all* vectors $\mathbf{u}$ and **v** in the vector space *V* and for any scalar *k* we have:
> (a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ [*T* preserves vector addition]
> (b) $T(k\mathbf{u}) = kT(\mathbf{u})$ [*T* preserves scalar multiplication]

Checking (a):

$$\begin{aligned} T(\mathbf{X}+\mathbf{Y}) &= \frac{1}{2}\left(\mathbf{A}(\mathbf{X}+\mathbf{Y}) + (\mathbf{X}+\mathbf{Y})\mathbf{A}\right) \\ &= \frac{1}{2}(\mathbf{AX}+\mathbf{AY}+\mathbf{XA}+\mathbf{YA}) \\ &= \underbrace{\frac{1}{2}(\mathbf{AX}+\mathbf{XA})}_{=T(\mathbf{X})} + \underbrace{\frac{1}{2}(\mathbf{AY}+\mathbf{YA})}_{=T(\mathbf{Y})} = T(\mathbf{X}) + T(\mathbf{Y}) \end{aligned}$$

We have $T(\mathbf{X}+\mathbf{Y}) = T(\mathbf{X}) + T(\mathbf{Y})$.

Checking (b):

$$\begin{aligned} T(k\mathbf{X}) &= \frac{1}{2}\left(\mathbf{A}(k\mathbf{X}) + (k\mathbf{X})\mathbf{A}\right) \\ &= k\underbrace{\frac{1}{2}(\mathbf{AX}+\mathbf{XA})}_{=T(\mathbf{X})} = kT(\mathbf{X}) \end{aligned}$$

We have $T(k\mathbf{X}) = kT(\mathbf{X})$.

By Definition (5.2) we conclude that the given transformation is linear.

2. (i) The dot product of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ is given by

$$T(\mathbf{e}_1)\cdot T(\mathbf{e}_2) = \begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ba = 0$$

The vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are orthogonal or the angle between them is $90°$.

(ii) The matrix is given by

$$\mathbf{A} = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(iii) We are given that $a^2 + b^2 = 1$ so the norm (length) of each of the vectors is

$$\|T(\mathbf{e}_1)\| = \|T(\mathbf{e}_2)\| = 1$$

This means that $\{T(\mathbf{e}_1)\ ,T(\mathbf{e}_2)\}$ are orthonormal vectors which implies that the matrix **A** is *orthogonal* so $\mathbf{A}^{-1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{T} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Hence $T^{-1}(\mathbf{x}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ -bx+ay \end{pmatrix}$.

3. Remember the kernel of *T* is the vector **x** which satisfies $T(\mathbf{x}) = \mathbf{0}$. Substituting the given vector **x** into $T(\mathbf{x}) = \mathbf{Ax}$:

$$T(\mathbf{x}) = \mathbf{Ax} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & -1 & -2 \\ 5 & -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Hence the given vector $\mathbf{x} = \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}^T$ is a member of the kernel of *T*.

We are given $\operatorname{rank}(T) = 3$ so by using the following result of chapter 5:

> Dimension Theorem (5.12). Let $T: V \to W$ be a linear transformation from an *n* dimensional vector space *V* to a vector space *W*. Then
> $$\operatorname{rank}(T) + \operatorname{nullity}(T) = n$$

We have $3 + \operatorname{nullity}(T) = 4 \quad \Rightarrow \quad \operatorname{nullity}(T) = 1$. *What does* $\operatorname{nullity}(T) = 1$ *mean?* There is one *non-zero* vector which is a basis for the kernel of *T*. Hence the given vector $\mathbf{x} = \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}^T$ is a basis for the kernel of *T*.

4. First we take the common scalar of *r* out of the matrix; $\mathbf{A} = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$.

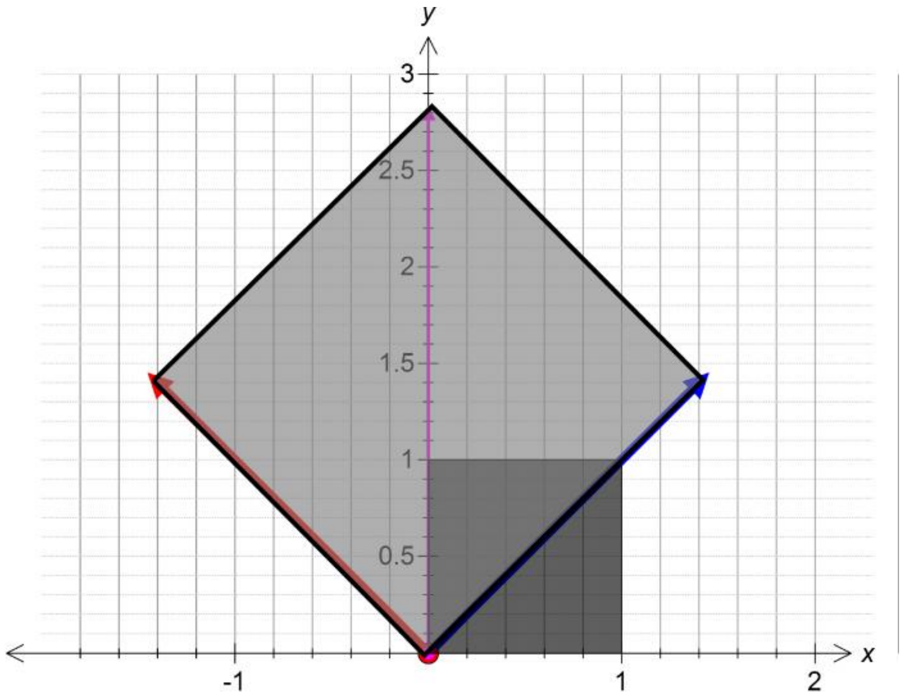
Substituting $r = 2,\ \theta = 45°$ into this matrix gives

$$\mathbf{A} = 2\begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = 2\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Applying this matrix to the given vectors $\mathbf{B} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$:

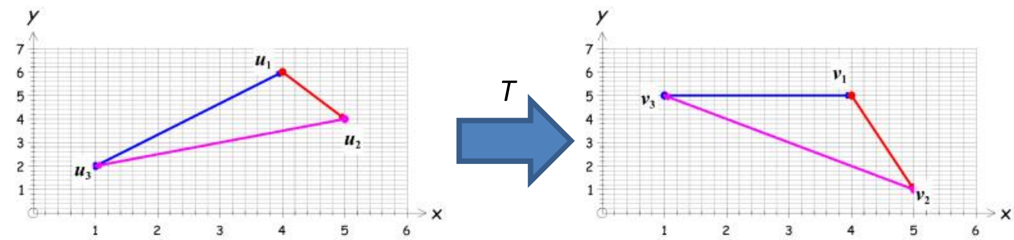
$$\mathbf{AB} = \sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Sketching this on the plane gives



The matrix **A** rotates the unit square by $45°$ in a counter clockwise direction with the centre origin. Also scales the square by a factor of $\sqrt{2}$.

5. We are given the transformation $T(\mathbf{u}_i) = \mathbf{v}_i$:



We have

$$T(\mathbf{u}_i) = \mathbf{v}_i = \mathbf{A}\mathbf{u}_i$$

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$T(\mathbf{u}_1) = T\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = T(\mathbf{v}_1)$$

$$T(\mathbf{u}_2) = T\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} = T(\mathbf{v}_2)$$

$$T(\mathbf{u}_3) = T\left[\begin{pmatrix} -4 \\ -2 \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = T(\mathbf{v}_3)$$

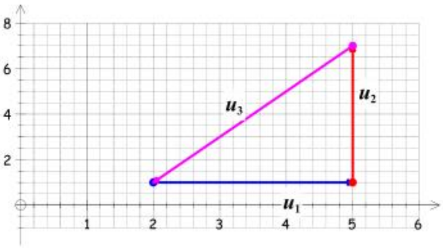
Solving these equations for the unknowns gives

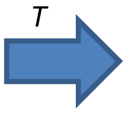
$$a = 1,\ b = 0,\ c = -\frac{8}{5},\ d = \frac{6}{5}$$

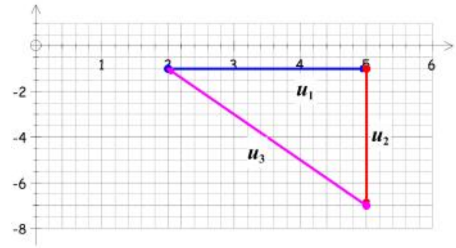
Hence our matrix **A** is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -8/5 & 6/5 \end{pmatrix}$$

6. We have







We are given that $T(\mathbf{u}_i) = \mathbf{v}_i = \mathbf{A}\mathbf{u}_i$ where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{matrix} & \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & 0 & -3 \\ 0 & 6 & -6 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -3 \\ 0 & -6 & 6 \end{pmatrix}$$

Solving these equations for the unknowns gives

$$a = 1,\ b = 0,\ c = 0,\ d = -1$$

Hence our matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

7. **A** is a 2 by 3 matrix so let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. Since **Ax** is a valid operation, **x** must be a 3 by 1 column vector, $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$:

$$\mathbf{Ax} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix}$$

The domain *U* of *T* is equal to $\mathbb{R}^3$ because we apply matrix **A** to the vector **x** which is an element of $\mathbb{R}^3$. The codomain *V* of *T* is $\mathbb{R}^2$ because the output vector $\mathbf{Ax}$ is an element of $\mathbb{R}^2$.

8. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + 2\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} a + 2b = 2 \\ c + 2d = 1 \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + 3\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} a + 3b = 3 \\ c + 3d = 1 \end{matrix}$$

Solving these equations gives $a = d = 0,\ b = c = 1$ so our matrix is

$$\mathbf{A}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

9. (i) Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard unit vectors of $\mathbb{R}^2$. Then

$$T(\mathbf{e}_1)=3\mathbf{e}_1=\begin{pmatrix}3\\ 0\end{pmatrix} \text{ and } T(\mathbf{e}_2)=3\mathbf{e}_2=\begin{pmatrix}0\\ 3\end{pmatrix}$$

The matrix which represents this transformation is

$$\left(T(\mathbf{e}_1) \quad T(\mathbf{e}_2)\right)=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$$

(ii) Similarly we have $\begin{pmatrix}n & 0\\ 0 & n\end{pmatrix}$.

(iii) Since the given transform scales the vector by 3 so inverse transform must scale it by 1/3. Similarly for *n* provided *n* is not zero. Hence we have

$$T^{-1}(\mathbf{x})=\frac{1}{3}\mathbf{x} \text{ and } T^{-1}(\mathbf{x})=\frac{1}{n}\mathbf{x} \text{ where } n\neq 0$$

If *n* is zero then the inverse transform T^{-1} does not exist.

10. Since *T* is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ so the matrix representing *T* is a 2 by 2. Let $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then using $T\left[\begin{pmatrix}1\\ 2\end{pmatrix}\right]=3\begin{pmatrix}1\\ 2\end{pmatrix}$ and $T\left[\begin{pmatrix}1\\ 4\end{pmatrix}\right]=-2\begin{pmatrix}1\\ 4\end{pmatrix}$ we have

$$T\left[\begin{pmatrix}1\\ 2\end{pmatrix}\right]=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}=\begin{pmatrix}3\\ 6\end{pmatrix} \Rightarrow \begin{matrix}a+2b=3\\ c+2d=6\end{matrix}$$

$$T\left[\begin{pmatrix}1\\ 2\end{pmatrix}\right]=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 4\end{pmatrix}=\begin{pmatrix}-2\\ -8\end{pmatrix} \Rightarrow \begin{matrix}a+4b=-2\\ c+4d=-8\end{matrix}$$

Solving these equations gives

$$a=8,\ b=-5/2,\ c=20,\ d=-7$$

The matrix $\mathbf{A}=\begin{pmatrix}8 & -5/2\\ 20 & -7\end{pmatrix}$.

11. Let $\mathbf{A}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}$ then

$$\mathbf{Ax}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}a_{11}\\ a_{21}\\ a_{31}\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

Similarly the second column vector of matrix **A** is $\begin{pmatrix}4\\5\\6\end{pmatrix}$. We have $\mathbf{A}=\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$.

How do we find the kernel of T?

It is the set of vectors **x** such that $\mathbf{Ax}=\mathbf{O}$. Using row operations we have

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\end{matrix}\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$$

Executing the following row operations

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3-3\mathrm{R}_1\end{matrix}\begin{pmatrix}1&4\\0&-3\\0&-6\end{pmatrix}$$

By carrying out appropriate row operations we have

$\mathbf{R}=\begin{pmatrix}1&4\\0&1\\0&0\end{pmatrix}$ and solving $\mathbf{Rx}=\mathbf{O}$ gives $x=y=0$

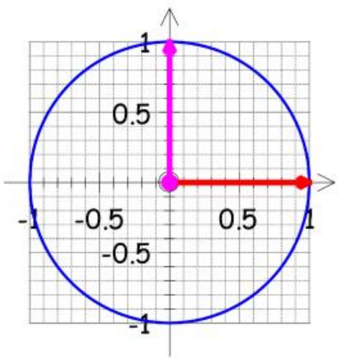
Hence the kernel of *T* is the zero vector $\{\mathbf{O}\}$.

12. (a) For $a=b=1$ we have

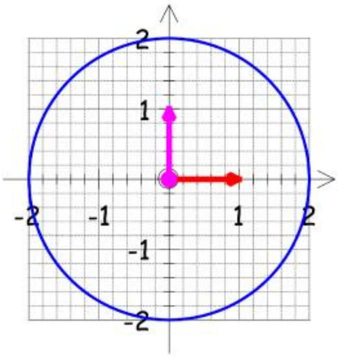
$$T\left[\begin{pmatrix}\cos(\text{¯})\\\sin(\text{¯})\end{pmatrix}\right]=\cos(\text{¯})\mathbf{e}_1+\sin(\text{¯})\mathbf{e}_2$$

$$=\cos(\text{¯})\begin{pmatrix}1\\0\end{pmatrix}+\sin(\text{¯})\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\cos(\text{¯})\\\sin(\text{¯})\end{pmatrix}$$

For different values of ¯ we have the unit circle:



(b) For $a=b=2$ we have circle with centre origin and radius 2:

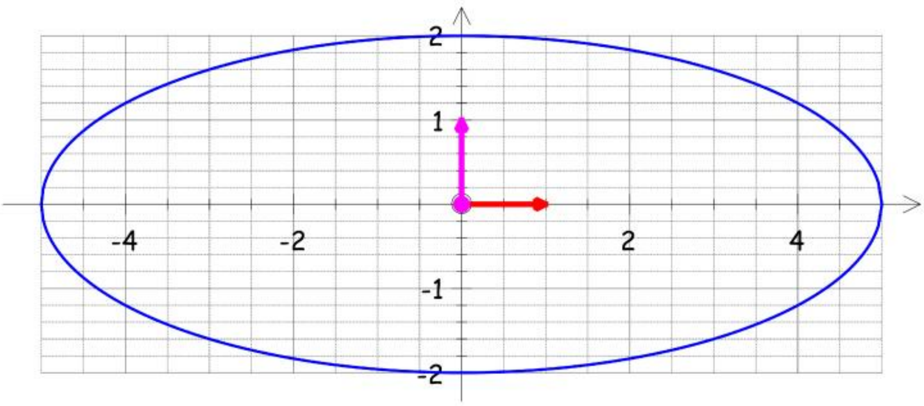


(c) For $a=5,\ b=2$ we have

$$T\begin{pmatrix}5\cos(\bar{\ })\\2\sin(\bar{\ })\end{pmatrix}=5\cos(\bar{\ })\mathbf{e}_1+2\sin(\bar{\ })\mathbf{e}_2$$

$$=5\cos(\bar{\ })\begin{pmatrix}1\\0\end{pmatrix}+2\sin(\bar{\ })\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}5\cos(\bar{\ })\\2\sin(\bar{\ })\end{pmatrix}$$

For various values of $\bar{\ }$ we have an ellipse:



13. We need to find the dot product of the given vectors **u** and **v**:

$$\mathbf{u}\cdot\mathbf{v}=\begin{pmatrix}\cos(\bar{\ })\\\sin(\bar{\ })\end{pmatrix}\cdot\begin{pmatrix}-\sin(\bar{\ })\\\cos(\bar{\ })\end{pmatrix}=-\cos(\bar{\ })\sin(\bar{\ })+\sin(\bar{\ })\cos(\bar{\ })=0$$

Vectors **u** and **v** are orthogonal. Need to check that these are unit vectors as well:

$$\|\mathbf{u}\|^2=\left\|\begin{pmatrix}\cos(\bar{\ })\\\sin(\bar{\ })\end{pmatrix}\right\|^2=\cos^2(\bar{\ })+\sin^2(\bar{\ })=1$$

Similarly $\|\mathbf{v}\|=1$. Hence **u** and **v** are orthonormal vectors.

(a) Evaluating $T(\mathbf{u})$ gives

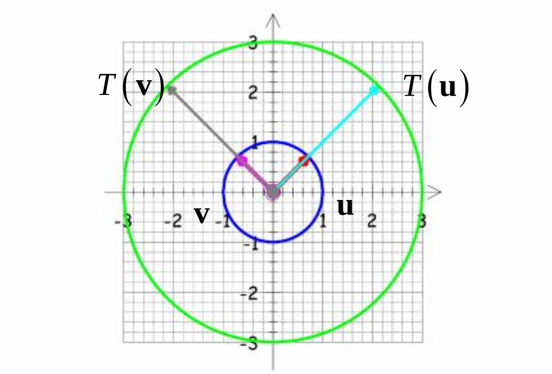
$$T(\mathbf{u})=\begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}\cos(\bar{\ })\\\sin(\bar{\ })\end{pmatrix}=\begin{pmatrix}3\cos(\bar{\ })\\3\sin(\bar{\ })\end{pmatrix}=3\mathbf{u}$$

Similarly

$$T(\mathbf{v})=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}-\sin(\bar{~})\\ \cos(\bar{~})\end{pmatrix}=\begin{pmatrix}-3\sin(\bar{~})\\ 3\cos(\bar{~})\end{pmatrix}=3\mathbf{v}$$

Since $3\mathbf{u}\cdot 3\mathbf{v}=9(\mathbf{u}\cdot\mathbf{v})=9(0)=0$.

Hence vectors $T(\mathbf{u})$ and $T(\mathbf{v})$ are orthogonal but not normalized (don't have a length of 1).



(b) Evaluating $T(\mathbf{u})$ gives

$$T(\mathbf{u})=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\cos(\bar{~})\\ \sin(\bar{~})\end{pmatrix}=\begin{pmatrix}3\cos(\bar{~})\\ 2\sin(\bar{~})\end{pmatrix}$$

Similarly we have

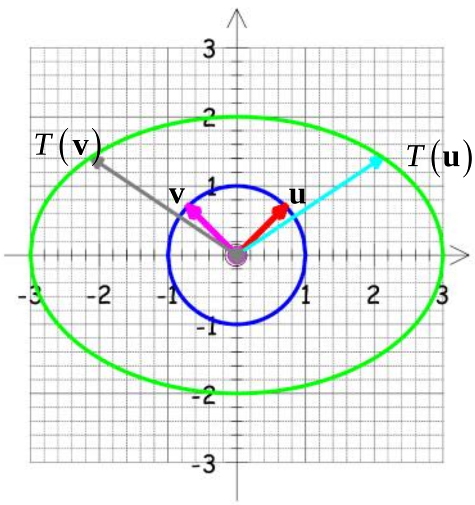
$$T(\mathbf{v})=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-\sin(\bar{~})\\ \cos(\bar{~})\end{pmatrix}=\begin{pmatrix}-3\sin(\bar{~})\\ 2\cos(\bar{~})\end{pmatrix}$$

Evaluating the dot product of these vectors gives

$$T(\mathbf{u})\cdot T(\mathbf{v})=\begin{pmatrix}3\cos(\bar{~})\\ 2\sin(\bar{~})\end{pmatrix}\cdot\begin{pmatrix}-3\sin(\bar{~})\\ 2\cos(\bar{~})\end{pmatrix}$$

$$=-9\cos(\bar{~})\sin(\bar{~})+4\cos(\bar{~})\sin(\bar{~})=-5\cos(\bar{~})\sin(\bar{~})\neq 0$$

These vectors are not orthogonal.



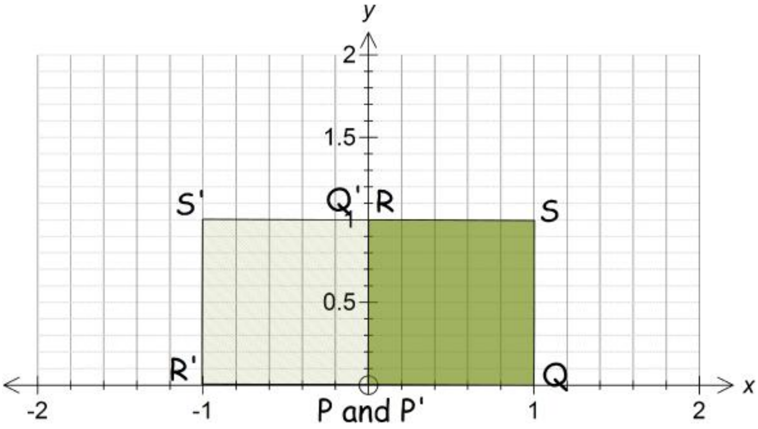
14. The unit square can be represented by coordinates of each vertex written as a column vector in a matrix:

$$\mathbf{B}=\begin{pmatrix} \overset{P}{0} & \overset{Q}{1} & \overset{R}{0} & \overset{S}{1} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) Applying the given matrix **A** to each of the column vectors in the matrix **B** gives

$$\mathbf{AB}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \overset{P}{0} & \overset{Q}{1} & \overset{R}{0} & \overset{S}{1} \\ 0 & 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} \overset{P'}{0} & \overset{Q'}{0} & \overset{R'}{-1} & \overset{S'}{-1} \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

We can sketch this

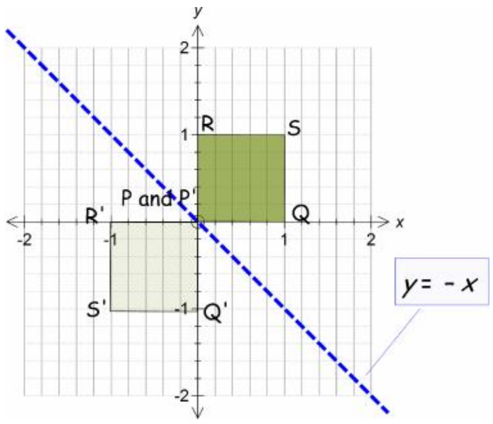


Rotate about the origin through $90°$ anti-clockwise.

(b) Similarly we have

$$\mathbf{AB}=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overset{P}{0} & \overset{Q}{1} & \overset{R}{0} & \overset{S}{1} \\ 0 & 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} \overset{P'}{0} & \overset{Q'}{0} & \overset{R'}{-1} & \overset{S'}{-1} \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

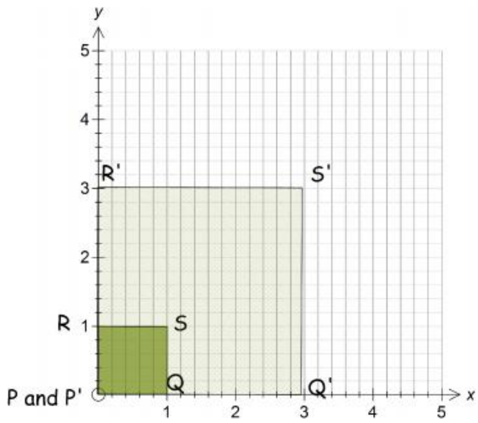
The sketch is



Reflection in the line $y=-x$.

(c) Also

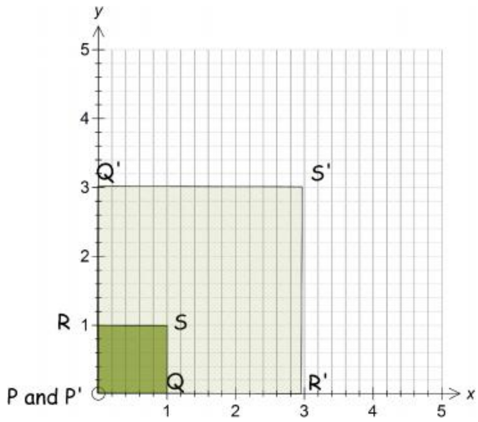
$$\mathbf{AB}=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$



Scaled by a factor of 3.

(d) Finally we have

$$\mathbf{AB} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 0 & 3 \end{pmatrix}$$



Reflected in the line $y = x$ and scaled by a factor of 3.

15. Substituting $^{-} = 45°$ into

$$T(\mathbf{x}) = M\begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix}\mathbf{x} + \mathbf{y} = \frac{M}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \mathbf{y}$$

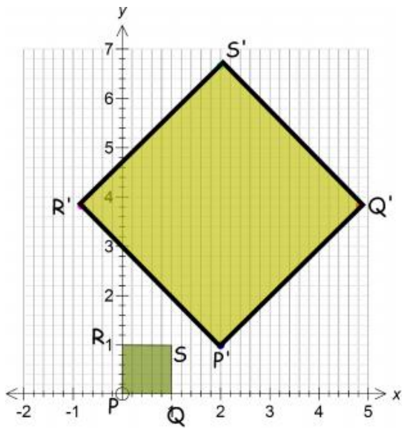
Let $\mathbf{B}$ be the matrix which represents each of the vertices of the unit square:

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) For $M = 4$, $\mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ we have

$$\frac{4}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\overset{\begin{matrix} P & Q & R & S \end{matrix}}{\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}} + \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \overset{\begin{matrix} P' & Q' & R' & S' \end{matrix}}{\begin{pmatrix} 2 & 4.83 & -0.83 & 2 \\ 1 & 3.83 & 3.83 & 6.66 \end{pmatrix}}$$

Sketching this gives:



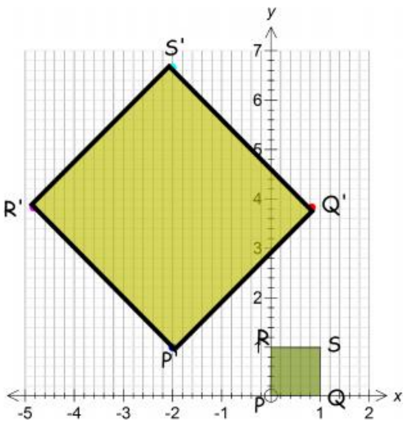
Firstly the square has been rotated by $45°$ in anti-clockwise direction with the centre the origin. Then it has been enlarged by a scale factor of 4 and shifted to the right by 2 units and up by 1 unit.

(b) For $M = 4$, $\mathbf{y} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ we have

$$\begin{matrix} & & P & Q & R & S & & & & & & P' & Q' & R' & S' \end{matrix}$$

$$\frac{4}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0.83 & -4.83 & -2 \\ 1 & 3.83 & 3.83 & 6.66 \end{pmatrix}$$

Sketching this gives:



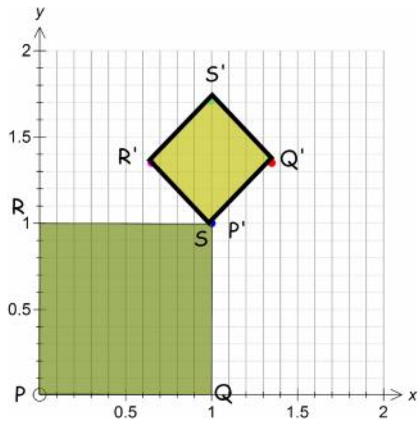
Firstly the square has been rotated by $45°$ in anti-clockwise direction with the centre the origin. Then it has been enlarged by a scale factor of 4 and shifted to the left by 2 units and up by 1 unit.

(c) For $M = \frac{1}{2}$, $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we have

$$\begin{matrix} & & P & Q & R & S & & & & & & P' & Q' & R' & S' \end{matrix}$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1.35 & 0.65 & 1 \\ 1 & 1.35 & 1.35 & 1.71 \end{pmatrix}$$

Sketching this gives:

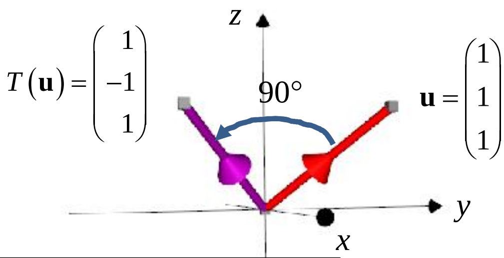


Firstly the square has been rotated by $45°$ in anti-clockwise direction with the centre the origin. Then it has been enlarged by a scale factor of 1/2 and shifted to the right and up by 1 unit.

16. (a) Applying the given transformation to the vector **u** with $\bar{} = 90°$ gives

$$\mathbf{M}_1\mathbf{u} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(90°) & -\sin(90°) \\ 0 & \sin(90°) & \cos(90°) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \cos(90°)-\sin(90°) \\ \sin(90°)+\cos(90°) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Sketching this on $x-y-z$ axes gives

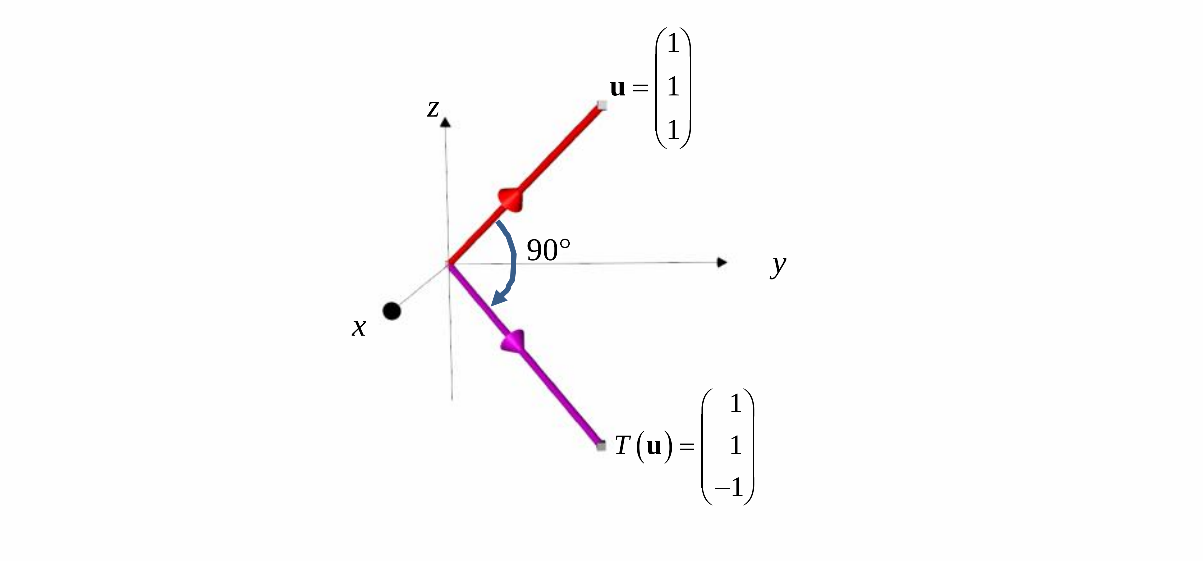


The linear operator T rotates the vector **u** by $90°$ about the $x-$axis.

(b) Similarly we have

$$\mathbf{M}_2\mathbf{u} = \begin{pmatrix} \cos(90°) & 0 & \sin(90°) \\ 0 & 1 & 0 \\ -\sin(90°) & 0 & \cos(90°) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(90°)+\sin(90°) \\ 1 \\ -\sin(90°)+\cos(90°) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Sketching this gives

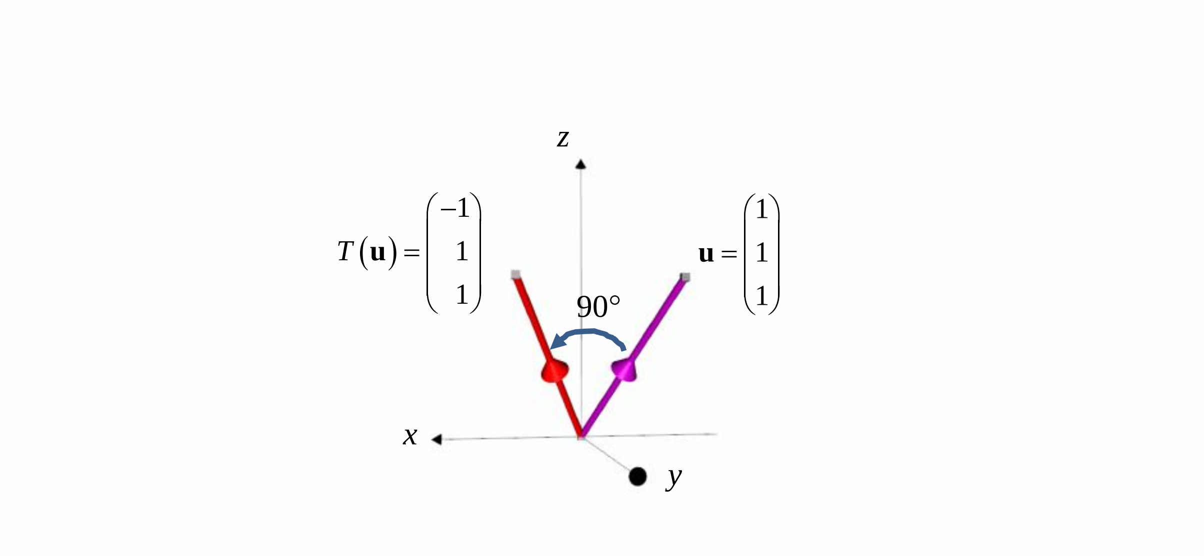


This time the linear operator T rotates the vector **u** by $90°$ about the y – axis.

(c) We have

$$\mathbf{M}_3\mathbf{u}=\begin{pmatrix}\cos(90°) & -\sin(90°) & 0\\ \sin(90°) & \cos(90°) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}\cos(90°)-\sin(90°)\\ \sin(90°)+\cos(90°)\\ 1\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Sketching this we have



The linear operator T rotates the vector **u** by $90°$ about the z – axis.

It is straightforward to show that $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are orthogonal matrices. *What is the inverse of an orthogonal matrix?*

$\mathbf{M}^{-1}=\mathbf{M}^T$. We have

$$\mathbf{M}_1^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\overline{\ }) & -\sin(\overline{\ })\\ 0 & \sin(\overline{\ }) & \cos(\overline{\ })\end{pmatrix}^T=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\overline{\ }) & \sin(\overline{\ })\\ 0 & -\sin(\overline{\ }) & \cos(\overline{\ })\end{pmatrix}$$

Therefore $T^{-1}(\mathbf{x})$ in this case is equal to

$$T^{-1}(\mathbf{x}) = \mathbf{M}_1^{-1}\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\bar{\ }) & \sin(\bar{\ }) \\ 0 & -\sin(\bar{\ }) & \cos(\bar{\ }) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y\cos(\bar{\ }) + z\sin(\bar{\ }) \\ -y\sin(\bar{\ }) + z\cos(\bar{\ }) \end{pmatrix}$$

For the matrix in part (b) we have

$$\mathbf{M}_2^{-1} = \begin{pmatrix} \cos(\bar{\ }) & 0 & \sin(\bar{\ }) \\ 0 & 1 & 0 \\ -\sin(\bar{\ }) & 0 & \cos(\bar{\ }) \end{pmatrix}^T = \begin{pmatrix} \cos(\bar{\ }) & 0 & -\sin(\bar{\ }) \\ 0 & 1 & 0 \\ \sin(\bar{\ }) & 0 & \cos(\bar{\ }) \end{pmatrix}$$

Hence

$$T^{-1}(\mathbf{x}) = \mathbf{M}_2^{-1}\mathbf{x} = \begin{pmatrix} \cos(\bar{\ }) & 0 & -\sin(\bar{\ }) \\ 0 & 1 & 0 \\ \sin(\bar{\ }) & 0 & \cos(\bar{\ }) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos(\bar{\ }) - z\sin(\bar{\ }) \\ y \\ x\sin(\bar{\ }) + z\cos(\bar{\ }) \end{pmatrix}$$

For the last matrix we have

$$\mathbf{M}_3^{-1} = \begin{pmatrix} \cos(\bar{\ }) & -\sin(\bar{\ }) & 0 \\ \sin(\bar{\ }) & \cos(\bar{\ }) & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \cos(\bar{\ }) & \sin(\bar{\ }) & 0 \\ -\sin(\bar{\ }) & \cos(\bar{\ }) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We have

$$T^{-1}(\mathbf{x}) = \mathbf{M}_3^{-1}\mathbf{x} = \begin{pmatrix} \cos(\bar{\ }) & \sin(\bar{\ }) & 0 \\ -\sin(\bar{\ }) & \cos(\bar{\ }) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos(\bar{\ }) + y\sin(\bar{\ }) \\ -x\sin(\bar{\ }) + y\cos(\bar{\ }) \\ z \end{pmatrix}$$

17. Let $\mathbf{u} = \mathbf{O}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then

$$T(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \mathbf{O} = 0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \mathbf{O} = T(\mathbf{y})$$

We have $T(\mathbf{x}) = T(\mathbf{y})$ but $\mathbf{x} \neq \mathbf{y}$.

18. *Proof*.

We are given that $T(\mathbf{u}) = \mathbf{x} \cdot \mathbf{u}$ and $T(\mathbf{u}) = \mathbf{y} \cdot \mathbf{u}$. Equating these gives

$$\mathbf{x} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u}$$
$$\mathbf{x} \cdot \mathbf{u} - \mathbf{y} \cdot \mathbf{u} = 0$$
$$(\mathbf{x} - \mathbf{y}) \cdot \mathbf{u} = 0$$

This means that $\mathbf{x} - \mathbf{y}$ and **u** are orthogonal where **u** is any vector **u** in $\mathbb{R}^n$. By

Proposition (4.10). Every vector in an inner product space *V* is orthogonal to the zero vector, **O**.

We must have $\mathbf{x} - \mathbf{y} = \mathbf{O} \Rightarrow \mathbf{x} = \mathbf{y}$.

■

19. Use the above definition (5.2) to show that the given transformation is linear.

What does kernel of T mean?

Those vectors in the start vector space which get transformed to the zero vector. We need to find $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ such that

$$T(\mathbf{x}) = x + y + z = 0$$

From this we have $x = -y - z$. Let $y = r,\ z = s$ where *r* and *s* are any real numbers then $x = -y - z = -r - s$. In vector form we have our solution

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -r-s \\ r \\ s \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the kernel is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

20. We use the following important result to find the rank and nullity of *T*:

> Dimension Theorem (5.12). Let $T: V \to W$ be a linear transformation from an *n* dimensional vector space *V* to a vector space *W*. Then
>
> $$\text{rank}(T) + \text{nullity}(T) = n$$

(a) Notice that the row vectors of the given matrix **A** are all multiples of each other so the rank of *T* is one. Applying the above result with $rank(T) = 1$ and $n = 4$ gives

$$1 + \text{nullity}(T) = 4 \quad \Rightarrow \quad \text{nullity}(T) = 3$$

The nullity of *T* gives us the dimension of the kernel of *T* which is 3 in our case. The kernel is the subspace of $\mathbb{R}^4$ which is transformed to the zero vector. Since each of the rows is a multiple of each other in matrix **A** so we need to solve

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} -2 & 5 & 7 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We have

$$-2x + 5y + 7z + 13w = 0$$

$$x = \frac{1}{2}(5y + 7z + 13w)$$

Our free variables are $y,\ z$ and w. Let $y=r,\ z=s$ and $w=t$ where these are any real numbers. Hence we have $x=\frac{1}{2}\left(5y+7z+13w\right)=\frac{1}{2}\left(5r+7s+13t\right)$. In vector form

$$\mathbf{x}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}\left(5r+7s+13t\right)/2\\r\\s\\t\end{pmatrix}=r\begin{pmatrix}5/2\\1\\0\\0\end{pmatrix}+s\begin{pmatrix}7/2\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}13/2\\0\\0\\1\end{pmatrix}$$

Hence a basis for kernel of *T* is $\left\{\begin{pmatrix}5/2\\1\\0\\0\end{pmatrix},\begin{pmatrix}7/2\\0\\1\\0\end{pmatrix},\begin{pmatrix}13/2\\0\\0\\1\end{pmatrix}\right\}$.

(b) Labelling the rows of the given matrix **A** we have

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\\\mathrm{R}_3\\\mathrm{R}_4\end{matrix}\begin{pmatrix}1&3&4&7\\2&-1&5&1\\5&2&1&1\\4&13&3&20\end{pmatrix}$$

Carrying out the row operations $\mathrm{R}_2-2\mathrm{R}_1$, $\mathrm{R}_3-5\mathrm{R}_1$ and $\mathrm{R}_4-4\mathrm{R}_1$:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^*=\mathrm{R}_2-2\mathrm{R}_1\\\mathrm{R}_3^*=\mathrm{R}_3-5\mathrm{R}_1\\\mathrm{R}_4^*=\mathrm{R}_4-4\mathrm{R}_1\end{matrix}\begin{pmatrix}1&3&4&7\\0&-7&-3&-13\\0&-13&-19&-34\\0&1&-13&-8\end{pmatrix}$$

Interchange rows R_2^* and R_4^*:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{**}\\\mathrm{R}_3^{*}\\\mathrm{R}_4^{**}\end{matrix}\begin{pmatrix}1&3&4&7\\0&1&-13&-8\\0&-13&-19&-34\\0&-7&-3&-13\end{pmatrix}$$

Executing the following row operations:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{**}\\\mathrm{R}_3^{**}=\mathrm{R}_3^{*}+13\mathrm{R}_2^{**}\\\mathrm{R}_4^{\dagger}=\mathrm{R}_4^{**}+7\mathrm{R}_2^{**}\end{matrix}\begin{pmatrix}1&3&4&7\\0&1&-13&-8\\0&0&-188&-138\\0&0&-94&-69\end{pmatrix}$$

The row operation we carry out is

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2^{**}\\\mathrm{R}_3^{**}\\2\mathrm{R}_4^{\dagger}-\mathrm{R}_3^{**}\end{matrix}\begin{pmatrix}1&3&4&7\\0&1&-13&-8\\0&0&-188&-138\\0&0&0&0\end{pmatrix}$$

Multiplying the second row by $-1/2$ gives

$$\begin{matrix} R_1 \\ R_2^{**} \\ R_3^{**} \\ 2R_4^{\dagger}-R_3^{**} \end{matrix} \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & -13 & -8 \\ 0 & 0 & 94 & 69 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the rank of *T* is 3 because we have 3 linearly independent rows. By using the above result $\text{rank}(T)+\text{nullity}(T)=n$ with $n=4$ and $\text{rank}(T)=3$ we have

$$3+\text{nullity}(T)=4 \quad \Rightarrow \quad \text{nullity}(T)=1$$

A basis for the kernel is the vectors which satisfy $\mathbf{Ax}=\mathbf{O}$. Using the last matrix:

$$\begin{matrix} x & y & z & w & \end{matrix}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 7 & 0 \\ 0 & 1 & -13 & -8 & 0 \\ 0 & 0 & 94 & 69 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

By the third row we have

$$94z+69w=0 \quad \Rightarrow \quad z=-\frac{69}{94}w$$

Let $w=-94r$ where *r* is any real number. Then $z=-\frac{69}{94}(-94r)=69r$.

By the second row

$$y-13z-8w=0 \quad \Rightarrow \quad y=13z+8w=13(69r)+8(-94r)=145r$$

From the top row we have

$$\begin{aligned} x+3y+4z+7w=0 \quad \Rightarrow \quad x &= -3y-4z-7w \\ &= -3(145r)-4(69r)-7(-94r)=-53r \end{aligned}$$

We have $x=-53r$, $y=145r$, $z=69r$ and $w=-94r$:

$$\mathbf{x}=\begin{pmatrix} -53 \\ 145 \\ 69 \\ -94 \end{pmatrix} r$$

Therefore a basis for the kernel is the vector $\left\{\begin{pmatrix} -53 \\ 145 \\ 69 \\ -94 \end{pmatrix}\right\}$.

21. The linearly independent columns in $\mathbf{A}=\begin{pmatrix}1&2&3&0&1&2&0\\0&1&4&0&2&4&0\\0&0&0&2&3&6&0\\0&0&0&0&0&8&3\end{pmatrix}$ are first, second, fourth and last because we can make the remaining columns from these and they all have a leading element. Hence our free variables are $x_3,\ \ x_5,\ x_6$. We need to solve $\mathbf{Ax}=\mathbf{O}$ because we are interested in the kernel of **A**. We have

$$\begin{matrix}x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \end{matrix}$$
$$\left(\begin{array}{ccccccc|c}1&2&3&0&1&2&0&0\\0&1&4&0&2&4&0&0\\0&0&0&2&3&6&0&0\\0&0&0&0&0&8&3&0\end{array}\right)$$

Using the bottom row we have

$$8x_6+3x_7=0 \quad\Rightarrow\quad x_7=-\frac{8}{3}x_6$$

Let $x_6=-3s$ then $x_7=8s$. Using the third row we have

$$2x_4+3x_5+6x_6=0 \;\Rightarrow\; x_4=\frac{-6x_6-3x_5}{2}=-3x_6-\frac{3}{2}x_5$$

Substituting $x_6=-3s$ and $x_5=2t$ into this gives

$$x_4=-3\left(-3s\right)-\frac{3}{2}\left(2t\right)=9s-3t$$

From the second row we have

$$x_2+4x_3+2x_5+4x_6=0 \quad\Rightarrow\quad x_2=-4x_3-2x_5-4x_6$$

Substituting $x_3=r$, $x_5=2t$ and $x_6=-3s$ into this yields

$$x_2=-4r-4t+12s$$

Using the top row we have

$$x_1+2x_2+3x_3+x_5+2x_6=0 \;\Rightarrow\; x_1=-2x_2-3x_3-x_5-2x_6$$

Substituting $x_2=-4r-4t+12s$, $x_3=r$, $x_5=2t$ and $x_6=-3s$ gives

$$\begin{aligned}x_1&=-2\left(-4r-4t+12s\right)-3r-\left(2t\right)-2\left(-3s\right)\\&=8r+8t-24s-3r-2t+6s=5r+6t-18s\end{aligned}$$

Our null space solution is given by

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{pmatrix}=\begin{pmatrix}5r+6t-18s\\-4r-4t+12s\\r\\9s-3t\\2t\\-3s\\8s\end{pmatrix}=s\begin{pmatrix}-18\\12\\0\\9\\0\\-3\\8\end{pmatrix}+t\begin{pmatrix}6\\-4\\0\\-3\\2\\0\\0\end{pmatrix}+r\begin{pmatrix}5\\-4\\1\\0\\0\\0\\0\end{pmatrix}$$

The basis for the kernel is

$$\left\{\begin{pmatrix}-18\\12\\0\\9\\0\\-3\\8\end{pmatrix},\begin{pmatrix}6\\-4\\0\\-3\\2\\0\\0\end{pmatrix},\begin{pmatrix}5\\-4\\1\\0\\0\\0\\0\end{pmatrix}\right\}$$

22. Finding the inverse by using row operations gives:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2\end{matrix}\left(\begin{array}{cc|cc}a&-b&1&0\\b&a&0&1\end{array}\right)$$

Carrying out the row operation:

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2{*}=a\mathrm{R}_2-b\mathrm{R}_1\end{matrix}\left(\begin{array}{cc|cc}a&-b&1&0\\0&a^2+b^2&-b&a\end{array}\right)\qquad (*)$$

Since $\mathbf{u}=\begin{pmatrix}a\\b\end{pmatrix},\mathbf{v}=\begin{pmatrix}-b\\a\end{pmatrix}$ are normalized vectors so their length is 1 which means

$$a^2+b^2=1 \qquad (\dagger)$$

Substituting this into (*) gives

$$\begin{matrix}\mathrm{R}_1\\\mathrm{R}_2{*}\end{matrix}\left(\begin{array}{cc|cc}a&-b&1&0\\0&1&-b&a\end{array}\right)$$

Executing the row operation

$$\begin{matrix}\mathrm{R}_1{*}=\mathrm{R}_1+b\mathrm{R}_2{*}\\\mathrm{R}_2{*}\end{matrix}\left(\begin{array}{cc|cc}a&0&1-b^2&ba\\0&1&-b&a\end{array}\right)$$

Multiplying the top row by $1/a$ gives

$$\begin{matrix}\mathrm{R}_1{*}/a\\\mathrm{R}_2{*}\end{matrix}\left(\begin{array}{cc|cc}1&0&\dfrac{1-b^2}{a}&\dfrac{ba}{a}\\0&1&-b&a\end{array}\right)$$

Using (†) and cancelling out the *a*'s gives

$$\begin{matrix} R_1^*/a \\ R_2^* \end{matrix}\left(\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & -b & a \end{array}\right)$$

Hence $T^{-1}\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

23. *How do we show* $T\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]=\begin{pmatrix} x+a \\ y+b \end{pmatrix}$ *is not a linear transformation?*

We can show that $T(\mathbf{O})\neq\mathbf{O}$ then by the below Proposition we conclude that *T* is not linear:

> Proposition (5.3). Let *V* and *W* be vector spaces and **u** and **v** be vectors in *V*. Let $T: V\to W$ be a linear transformation then we have the following:
> (a) $T(\mathbf{O})=\mathbf{O}$ where **O** is the zero vector.

Substituting $x=y=0$ into the given transformation

$$T\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]=\begin{pmatrix} 0+a \\ 0+b \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}\neq\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\mathbf{O}\quad\left[\text{Because } a\neq 0 \quad\text{or}\quad b\neq 0\right]$$

If $a=b=0$ then we have

$$T\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]=\begin{pmatrix} x \\ y \end{pmatrix}$$

This is the identity linear transformation.

24. From chapter 5 we have

$$\left(T(\mathbf{e}_1)\quad T(\mathbf{e}_2)\quad T(\mathbf{e}_3)\quad T(\mathbf{e}_4)\right)=\mathbf{A}=\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

(a) Therefore the matrix with respect to $(\mathbf{e}_1,\ \mathbf{e}_3,\ \mathbf{e}_2,\ \mathbf{e}_4)$ is given by swapping the second and third column vectors of matrix **A**:

$$\left(T(\mathbf{e}_1)\quad T(\mathbf{e}_3)\quad T(\mathbf{e}_2)\quad T(\mathbf{e}_4)\right)=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix}$$

(b) This time we need to find the matrix with respect to the basis

$$\left(\mathbf{e}_1,\ \mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\right)$$

Each of these vectors are given by

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \mathbf{e}_1+\mathbf{e}_2 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix},\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Applying the given matrix **A** to each of these vectors gives

$$T(\mathbf{e}_1) = \begin{pmatrix}1&2&0&1\\3&0&-1&2\\2&5&3&1\\1&2&1&3\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\3\\2\\1\end{pmatrix},\ T(\mathbf{e}_1+\mathbf{e}_2) = \begin{pmatrix}1&2&0&1\\3&0&-1&2\\2&5&3&1\\1&2&1&3\end{pmatrix}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\3\\7\\3\end{pmatrix}$$

$$T(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) = \begin{pmatrix}1&2&0&1\\3&0&-1&2\\2&5&3&1\\1&2&1&3\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\\10\\4\end{pmatrix},\ T(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4) = \begin{pmatrix}1&2&0&1\\3&0&-1&2\\2&5&3&1\\1&2&1&3\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\4\\11\\7\end{pmatrix}$$

Hence the matrix with reference to the given basis is

$$\big(T(\mathbf{e}_1)\quad T(\mathbf{e}_1+\mathbf{e}_2)\quad T(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)\quad T(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)\big) = \begin{pmatrix}1&3&3&4\\3&3&2&4\\2&7&10&11\\1&3&4&7\end{pmatrix}$$

25. We have

$$T(ax+b) = a\left(1+x+x^2\right)+b$$

By applying this linear transformation to each of the vectors in $B=(1,\ x)$ gives

$$T(1) = 0\left(1+x+x^2\right)+1 = 1 \qquad [a=0,\ b=1]$$

$$T(x) = 1\left(1+x+x^2\right)+0 = 1+x+x^2 \qquad [a=1,\ b=0]$$

We need to write each of these as the coordinates of the basis $C=\left(1,\ x,\ x^2\right)$:

$$T(1) = 1 = 1(1)+0(x)+0\left(x^2\right)$$

$$T(x) = 1+x+x^2 = 1(1)+1(x)+1\left(x^2\right)$$

The matrix **A** is given by $\mathbf{A} = \left(\left[T(1)\right]_C\quad \left[T(x)\right]_C\right) = \begin{pmatrix}1&1\\0&1\\0&1\end{pmatrix}$.

The coordinates of $\mathbf{p}=x+2$ with respect to the basis $B=(1,\ x)$ is

$$[\mathbf{p}]_B = \begin{pmatrix}2\\1\end{pmatrix}$$

We have $\left[T(\mathbf{p})\right]_C = \mathbf{A}[\mathbf{p}]_B = \begin{pmatrix}1&1\\0&1\\0&1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$. Hence we have

$$T(x+2)=3+x+x^2$$

26. By applying the given linear transformation to each of the vectors in

$$B=\{\sin(x)\cos(x),\ \sin^2(x),\ \cos^2(x)\}$$

Gives

$$\begin{aligned}T(\sin(x)\cos(x))&=\cos(x)\cos(x)-\sin(x)\sin(x)\\&=\cos^2(x)-\sin^2(x)\\&=0\sin(x)\cos(x)+(-1)\sin^2(x)+(1)\cos^2(x)\end{aligned}$$

Applying the transformation to the remaining two vectors

$$T(\sin^2(x))=2\sin(x)\cos(x)=2\sin(x)\cos(x)+0\sin^2(x)+0\cos^2(x)$$
$$T(\cos^2(x))=-2\cos(x)\sin(x)=-2\sin(x)\cos(x)+0\sin^2(x)+0\cos^2(x)$$

Let **A** be the matrix which represents T then

$$\begin{aligned}\mathbf{A}&=\left(\left[T(\sin(x)\cos(x))\right]_B\ \ \left[T(\sin^2(x))\right]_B\ \ \left[T(\cos^2(x))\right]_B\right)\\&=\begin{pmatrix}0&2&-2\\-1&0&0\\1&0&0\end{pmatrix}\end{aligned}$$

We need to find the derivative of $f(x)=-5\sin(x)\cos(x)+6\sin^2(x)-2\cos^2(x)$ by using this matrix. *What vector represents this function with respect to the given basis?*

$$\left[f(x)\right]_B=\begin{pmatrix}-5\\6\\-2\end{pmatrix}$$

We have

$$\mathbf{A}\left[f(x)\right]_B=\begin{pmatrix}0&2&-2\\-1&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}-5\\6\\-2\end{pmatrix}=\begin{pmatrix}16\\5\\-5\end{pmatrix}$$

Hence $f'(x)=16\sin(x)\cos(x)+5\sin^2(x)-5\cos^2(x)$.

27. *Proof.*

Consider the vectors

$$\mathbf{u},\ T(\mathbf{u}),\ T^2(\mathbf{u}),\ T^3(\mathbf{u}),\ \cdots,\ T^{n-1}(\mathbf{u}) \qquad (\dagger)$$

Suppose one of the vectors in this list is zero, $T^j(\mathbf{u})=\mathbf{O}$ where $0\le j<n$. Since T is a linear transformation so

$$T^n(\mathbf{u})=T^{n-j}T^j(\mathbf{u})=T^{n-j}\left[T^j(\mathbf{u})\right]=T^{n-j}[\mathbf{O}] \qquad (*)$$

By the following proposition of chapter 5:

Proposition (5.3). Let $T: V \to W$ be a linear transformation then
(a) $T(\mathbf{O}) = \mathbf{O}$ where **O** is the zero vector.

We have $T^{n-j}[\mathbf{O}] = T^{n-j-1}[T(\mathbf{O})] = \mathbf{O}$. Putting this into (*) gives $T^n(\mathbf{u}) = T^{n-j}[\mathbf{O}] = \mathbf{O}$. This is a contradiction because we are given $T^n(\mathbf{u}) \neq \mathbf{O}$. Hence none of the vectors in the list (†) are zero.

■